Bulk viscosity of QCD matter near the critical temperature

Kirill Tuchin in collaboration with D. Kharzeev

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3. Since N=4 SUSY YM is exactly conformally invariant the corresponding matter has vanishing bulk viscosity \( \zeta = 0 \). However, this is not necessarily true for QCD matter which conformal invariance is broken by quantum fluctuations.

4. Fortunately, we can determine a non-perturbative QCD contribution to the bulk viscosity \( \zeta \) without invoking any exotic theories.
Kubo formula for bulk viscosity

\[
\eta(\omega) \left( \delta_{il} \delta_{km} + \delta_{im} \delta_{kl} - \frac{2}{3} \delta_{ik} \delta_{lm} \right) + \zeta(\omega) \delta_{ik} \delta_{lm} = \frac{1}{\omega} \lim_{k \to 0} \int \int_0^\infty e^{i(\omega t - kr)} \langle [\theta_{ik}(t, r), \theta_{lm}(0)] \rangle dt d^3 x
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where \( \theta_{ik}(x) \) is the operator of the stress tensor.
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Contracting \( i,k \) and \( l,m \) (\( i=1,2,3 \)) we get in the static limit

\[ \zeta = \frac{1}{9} \lim_{\omega \to 0} \frac{1}{\omega} \int_0^\infty dt \int d3r \, e^{i\omega t} \langle [\theta_{ii}(x), \theta_{kk}(0)] \rangle \]
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\]

Indeed \( \langle [\int d^3x \theta_{00}, O] \rangle_{eq} = \langle [H, O] \rangle_{eq} = i \left\langle \frac{\partial O}{\partial t} \right\rangle_{eq} = 0 \)
Euclidean Green’s function

- It is convenient to use the Euclidean Green’s function. First, introduce the spectral density $\rho(\omega, \vec{p}) = -\frac{1}{\pi} \text{Im} G^R(\omega, \vec{p})$
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  G^R(\omega, \vec{p}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} G^R(u, \vec{p})}{u - \omega - i\epsilon} \, du = \int_{-\infty}^{\infty} \frac{\rho(u, \vec{p})}{\omega - u + i\epsilon} \, du
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- The retarded Green's function \( G^R(\omega, \vec{p}) \) of a bosonic excitation is related to the Euclidean Green's function \( G^E(\omega, \vec{p}) \) by analytic continuation
  \[ G^E(\omega, \vec{p}) = -G^R(i\omega, \vec{p}), \quad \omega > 0 \]
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We will calculate this object in QCD
Low-energy theorems (LET) in vacuum

• Conformal symmetry of QCD (@ m=0) is broken by vacuum fluctuations. However, there is still a certain unbroken symmetry which manifests itself as a set of LET. Consider an operator of canonical dimension $d$:

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\langle O \rangle_v \sim \left[ M_0 e^{-\frac{8\pi}{b g^2(\mu)}} \right]^d
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(by RGE)
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Novikov, Shifman, Vainshtein, Zakharov 1981
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• On the other hand, \( \langle O \rangle \) can be represented as a functional integral:

\[ \langle O \rangle_v = \int \mathcal{D}\tilde{A}_a^\mu O \exp \left( -i \frac{1}{4g^2} \int d^4x \tilde{F}_\mu^a \tilde{F}^a_{\mu\nu} \right) \]  

\( \tilde{F} = gF \)
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• Coupling \( g \) enters the lagrangian of Gluodynamics only as a pre-factor. Thus, differentiating with respect to \((-1/4 g^2)\) we get

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i \int dx \langle T[O(x), \tilde{F}^2(0)] \rangle = -\frac{d}{d(-1/4g^2)} \langle O \rangle_v
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\[ i \lim_{q \to 0} \int dx \ e^{iqx} \langle T[O(x), \frac{\beta(\alpha_s)}{4\alpha_s}F^2(0)] \rangle = (-d)\langle O \rangle_v \]
LET in vacuum (cont.)

\[ i \lim_{q \rightarrow 0} \int dx \, e^{iqx} \langle T[O(x), \frac{\beta(\alpha_s)}{4\alpha_s} F^2(0)] \rangle = (-d)(O)_v \]

- Using the trace of energy-momentum tensor for O:

\[ \theta^\mu_\mu = \frac{\beta(g)}{2g} F_\mu^a F^{a\mu} \]
LET in vacuum (cont.)

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- we derive

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i \int dx \langle T \theta^\mu_\mu(x), \theta^\mu_\nu(0) \rangle_{\text{connected}} = \langle \theta^\mu_\mu(0) \rangle (-4)
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- Differentiating n times we can derive LET for Green's function of n'th order.
  \[
  i^n \int dx_1 \ldots dx_n \left\langle T \theta^\mu_\mu_1(x_1), \ldots, \theta^\mu_n_\mu_n(x_n), \theta^\nu_\nu(0) \right\rangle_{\text{connected}} = \langle \theta^\mu_\mu(0) \rangle (-4)^n
  \]
LET in vacuum (cont.)

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Note: Coupling constant does not explicitly appear in LET \rightarrow

LET contain a non-perturbative information about the correlation functions.
Effective Dilaton Lagrangian

- LET can be saturated by a single scalar field $\chi$

$$L = \frac{|\epsilon_v|}{m^2} \frac{1}{2} e^{\chi/2} (\partial_\mu \chi)^2 + |\epsilon_v| e^\chi (1 - \chi)$$

$$\theta^\mu_\mu = -4 |\epsilon_v| e^\chi$$

- The field $\chi$ is referred to as the *dilaton*. In gluodynamics it corresponds to the scalar glueball. In the real world, it mixes up with light quarks to produce the $\sigma$-meson.

*Migdal, Shifman, 1982*
LET at finite temperature

- At finite temperature there is an additional dimensional parameter $T$. The grand potential $\Omega$ per unit volume can be written in imaginary time formalism as

$$\Omega = -T \ln Z = -T \ln \int \mathcal{D}\tilde{A}_a^\mu \exp \left(-\frac{1}{4g^2} \int_0^{1/T} d\tau \int d^3x \ \tilde{F}_{\mu\nu}^2 \tilde{F}^{\alpha\mu\nu} \right)$$
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• Dimensional analysis: $\langle O \rangle \sim \Lambda^d f(\Lambda/T)$ with $\Lambda \sim M_0 e^{-\frac{8\pi}{b g^2(\mu)}}$

• Differentiating with respect to $(-1/4 g^2)$ we obtain

$$\left(T \frac{\partial}{\partial T} - d\right)^n \langle O \rangle = \int_0^{1/T} d\tau_n \int d^3x_n \ldots \int_0^{1/T} d\tau_1 \int d^3x_1 \langle \theta_{\mu_n}^{\mu_n}(\tau_n, x_n) \ldots \theta_{\mu_1}^{\mu_1}(\tau_1, x_1) O(0, 0) \rangle_{\text{connected}}$$
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• In particular,

$$\int_0^{1/T} d\tau \int d^3 x \langle \theta^\mu(x), \theta^\nu(0) \rangle_{\text{connected}} = \left( T \frac{\partial}{\partial T} - 4 \right) \langle \theta^\mu(0) \rangle$$
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$$G^E(\omega \rightarrow 0, \vec{0}) = 2 \int^\infty_0 \frac{\rho(u, \vec{0})}{u} du$$

$$\mathcal{E} - 3P$$
Sum rule for the spectral density

Kharzeev, KT 2007

Note, that on the lattice one computes not $<\theta_{\mu\mu}>_T$ but, $<\theta_{\mu\mu}>_T - <\theta_{\mu\mu}>_0$ (subtracting the vacuum expectation value), i.e.

$$ (E - 3P)_{LAT} = \langle \theta^\mu_\mu \rangle_T - \langle \theta^\mu_\mu \rangle_0 $$

Denote $\langle \theta^\mu_\mu \rangle_0 = -4|\epsilon_v|$
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- The following exact sum rule holds

$$2 \int_0^\infty \frac{\rho(u, \vec{0})}{u} du = - \left(4 - T \frac{\partial}{\partial T}\right) \langle \theta \rangle_T = T^5 \frac{\partial}{\partial T} \frac{(\mathcal{E} - 3P)_{\text{LAT}}}{T^4} + 16|\epsilon_v|$$
Extracting the bulk viscosity

• In order to extract bulk viscosity we need an ansatz for the spectral density $\rho$

  \[ \rho(\omega) \sim \alpha_s^2 \omega^4. \] This divergent part is subtracted on both sides of the sum rule.

  At small frequencies we assume the following functional form which is odd in $\omega$ and has correct $\omega \to 0$ limit:

  \[ \rho(\omega, \vec{0}) = \frac{9 \zeta}{\pi} \frac{\omega^2}{\omega_0^2 + \omega^2} \]

• We have

  \[ \zeta = \frac{1}{9 \omega_0} \left\{ T^5 \frac{\partial}{\partial T} \frac{(\mathcal{E} - 3P)_{\text{LAT}}}{T^4} + 16|\epsilon_v| \right\} \]
Extracting the bulk viscosity (cont.)

- Parameter $\omega_0$ is a scale at which the perturbation theory becomes valid.

- In the region $1 < T/T_c < 3$ we find $\omega_0 \approx (T/T_c) 1.4$ GeV

- $T_c = 0.28$ GeV; $|\epsilon_v| = 0.62$ $T_c^4$. 

Kaczmarek, Karsch, Zantow, Petreczky, 2004
Lattice data

Boyd et al (Bielefeld), 1996

\[(\epsilon - 3\rho)/T^4\]

- \(\times 16^3\times 4\)
- \(\square 32^3\times 6\)
- \(\diamond 32^3\times 8\)
Bulk viscosity from the lattice

This result in a qualitative agreement with the recent lattice calculation.

Meyer, 2007
Bulk viscosity from the lattice

Bulk viscosity is

• small at $T \gg T_c$ in accord with expectations from pQCD.

• small at $T \ll T_c$ due to a derivative interactions

$$\theta_\mu = -\partial_\mu \pi^a \partial^\mu \pi^a + 2m_\pi^2 \pi^a \pi^a + \cdots$$

• large at $T \approx T_c$ where it becomes the dominant correction to the ideal hydrodynamics.

see also Paech, Pratt, 2006
Implications
Bulk viscosity and relaxation processes

- In general, pressure in a moving gas or liquid $P$ is different from the one in a static case $P_0$. Assuming that the deviation is small and noting that $P$ is scalar we can write

$$P = P_0 - \zeta \vec{\nabla} \cdot \vec{v}$$

$\zeta$ characterizes dependence of the forces in the medium on divergence of $v$, while $\eta$ characterizes forces depending on direction of $v$ and its gradient.
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  \( \zeta \) characterizes dependence of the forces in the medium on divergence of $\vec{v}$, while $\eta$ characterizes forces depending on direction of $\vec{v}$ and its gradient.

- Continuity equation implies that $\zeta$ describes dependence of pressure on the rate of density change.
  \[ \nabla \cdot \vec{v} = -\frac{1}{\rho} \frac{d\rho}{dt} \]
Bulk viscosity and relaxation processes

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$$P = P_0 - \zeta \vec{\nabla} \cdot \vec{v}$$

$\zeta$ characterizes dependence of the forces in the medium on divergence of $v$, while $\eta$ characterizes forces depending on direction of $v$ and its gradient.

• Continuity equation implies that $\zeta$ describes dependence of pressure on the rate of density change.

$$\vec{\nabla} \cdot \vec{v} = -\frac{1}{\rho} \frac{d\rho}{dt}$$

• If a system contains degrees of freedom which cannot be easily excited, then the pressure cannot follow the rapid change in density and is different from the equilibrium value $P_0$. Large $\zeta \rightarrow$ large $P - P_0$. 


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• Large deviation from equilibrium implies generation of a large amount of entropy: energy is dissipated in the relaxation process.
Relaxation time $\tau$

- All relaxation processes are characterized by a common asymptotic form of time-dependence

$$\frac{dN}{dt} = \frac{N_0 - N}{\tau} \Rightarrow N(t) = N_{in} e^{-t/\tau} + N_0(1 - e^{-t/\tau})$$

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$$P - P_0 = \frac{\tau \rho}{1 - i \omega \tau} (c_0^2 - c_\infty^2) \vec{\nabla} \cdot \vec{v}$$

where

$$c_0^2 = \left( \frac{\partial p}{\partial \rho} \right)_\text{eq} \quad c_\infty^2 = \left( \frac{\partial p}{\partial \rho} \right)_N$$
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- It follows that

$$\zeta = \frac{\tau \mathcal{E}}{1 - i\omega \tau} (c_\infty^2 - c_0^2)$$
Sound propagation

- Consider propagation of a sound wave of frequency $\omega$ and wave vector $k=\omega/c$, where $c^2=(\partial P/\partial \rho)$ and $P=P(\rho;\omega,\tau)$.

$$k = \omega \sqrt{\frac{1 - i\omega\tau}{c_0^2 - c_\infty^2 i\omega\tau}}$$
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At small frequencies $\omega \tau \ll 1$ sound propagation is adiabatic with

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★ In relativistic medium $c_\infty=1/\sqrt{3}$ (no interactions)
Relaxation time

- At $\omega \to 0$ (static, adiabatic case) we can use the lattice data to determine the relaxation time.

- **Lessons:**
  1. At $T \approx T_c$ relaxation processes are very slow.
  2. The system is far from equilibrium.
  3. Speed of sound is $c \approx c_\infty = 1/\sqrt{3} >> c_0$. 

\[ \frac{\tau}{\text{fm}} \]

\[ \begin{array}{c}
0.0 & 0.5 & 1.0 & 1.5 \\
0.9 & 1.0 & 1.1 & 1.2 & 1.3 & 1.4 & 1.5 \\
\end{array} \]

\[ \frac{T}{T_c} \]
Dilaton excitations in QGP

1. We have demonstrated that existence of a colorless scalar excitation of the trace of energy-momentum tensor (dilaton) is a very important feature of QGP near $T_c$.

2. Unlike in vacuum where the dilaton is massive (it is a part of the scalar glueball), at finite $T$ it becomes massless.
Propogation of a jet through QGP (A toy model)

- A jet propagating through the medium generates a dilaton sound wave in its wake. This is a shock wave of finite thickness $\sim \tau c_\infty = \tau/\sqrt{3}$. 
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$V < C$
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Summary

- We derived an exact sum rule for the spectral density of $\theta_{\mu\mu}$ correlator which relates it to $E-3P$ computed on the lattice.

- We used it to estimate the bulk viscosity in gluodynamics and found it to be large near $T=T_c$.

- A (small) contribution from light quarks will soon be calculated.

- Large $\zeta$ implies existence of a massless colorless scalar excitation of QGP $\Rightarrow$ important for energy loss, Mach cone etc.
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Work in progress!