Duality and Quantum Hall Systems

Duality in Your Everyday Life

with Rim Dib, Brian Dolan and C.A. Lutken
Outline

- Motivation and Preliminaries
  - The virtue of robust low-energy explanations
  - What is the Quantum Hall Effect?
- What Needs Explaining
  - Transitions among Quantum Hall Plateaux
- A phenomenological explanation
  - A symmetry interpretation: the group $\Gamma_0(2)$
- From whence the symmetry?
  - Statistics shifts and particle/vortex duality
- Spinoffs
  - New Predictions: Bosons and $\Gamma_0(2)$ Invariance
Why Stay Awake?

✓ **Field theory:**
  ✓ Many of the most interesting theories have recently been found to enjoy duality symmetries.
    ✓ *eg:* Electromagnetism with charges and monopoles has symmetry under their interchange together with $e$ replaced by $2\delta/e$.
  ✓ Quantum Hall Systems are the first laboratory systems with this kind of symmetry.
  ✓ May learn about new kinds of duality.

✓ **Condensed matter physics:**
  ✓ Duality helps identify robust low-energy features
    ✓ *eg* why are some QH features so accurate?
  ✓ Suggests existence of new kinds of Quantum Hall Systems
What is the Quantum Hall Effect?
Electrons/holes can be trapped on a two-dimensional surface using an electrostatic potential well.

- Occurs in real semiconducting devices.
- Knobs: $T$, $B$, $n$
  - Carrier densities may be adjusted by varying a gate voltage
The Conductivity Tensor

- Crossed electric and magnetic fields, or anisotropy, give currents not parallel to applied E fields
  - Described, for weak fields, by a conductivity tensor, $\sigma_{kl}$, or a resistivity tensor, $\rho_{kl} = (\sigma^{-1})_{kl}$

$$J_k = \sigma_{kl} E_l$$

- $\sigma_{kl} = \sigma_{lk}$: Ohmic conductivity
- $\sigma_{kl} = - \sigma_{lk}$: Hall conductivity
Isotropic 2d Systems

- Special Case: Isotropic Systems:
  - Choose: $E = E \mathbf{e}_x$, $B = B \mathbf{e}_z$
  - \[
  \begin{pmatrix}
  \sigma_{kl} \\
  \end{pmatrix} = \begin{pmatrix}
  \sigma_{xx} & \sigma_{xy} \\
  -\sigma_{xy} & \sigma_{xx} \\
  \end{pmatrix}
  \]

- Alternatively, in the upper-half complex plane:
  - $\sigma := \sigma_{xy} + i \sigma_{xx}$
  - Resistivity: $\rho := \rho_{xy} + i \rho_{xx} = -1/\sigma$
  - $\rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2}$
  - $\rho_{xy} = \frac{-\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2}$
Classical Hall Effect

- Lorentz force on moving charges induces a transverse current.
- Implies $\rho_{xy}$ proportional to $B$:

$$\rho_{xx} = \rho_0; \quad \rho_{xy} = \frac{h}{v e^2}$$

Where $\rho_0 = m/(n e^2 \tau_0)$; $v = n/(eB)$. 
Free-electron Quantum Description

- Free electrons in magnetic fields form evenly-spaced simple-harmonic Landau levels (LL), labelled by $n, k$
  - Energy: $\omega_e = \frac{eB}{m}$, Size: $1/\lambda^2 = eB$, Centre: $y_c = \lambda^2 k$
- Each LL is extremely degenerate in absence of interactions.
  - Number of available states proportional to sample area.
- Fraction of LL filled with N electrons is $\nu = n/eB$. 
Experiments:

- Experiment: at low temperatures $\rho_{xy}$ is **not** proportional to $B$.
- Plateaux with $\sigma_{xy} = ke^2/h$, with $k$ an odd-denominator fraction
  - Quantized with extraordinary accuracy!

Stormer: *Physica B* 177 (92) 401
A Quantum Hall Puzzle

- Why should $\sigma_{xx}$ vanish and $\sigma_{xy}$ be so accurately quantized in units of $e^2/h$?
- A microscopic picture:
  - Interactions and disorder broaden bands and localize electrons.
  - Conductivity independent of filling requires most electrons in band to be localized, so as not to conduct.
  - Narrow energy range in each band containing extended states causes jumps between plateaux.
Quantization: The Corbino Geometry

Laughlin’s gauge-invariance argument: Laughlin, PRB 23 5632

- Insertion of magnetic flux induces radial charge transfer
- One flux quantum gives a ‘large’ gauge transformation.
  - Induces level shift with transfer of a charge quantum

\[
\sigma_{xy} = \frac{\Delta Q}{\Delta \phi / \phi_0}
\]

Girvin, cond-mat/9907002
Other Experiments: Flow Between Plateaux

1. Semicircle Law as B is varied.
2. Transitions between which plateaux?
3. Universality of Critical Points
4. Superuniversality of Critical Exponents
5. $\rho_{xx} \diamond 1/\rho_{xx}$ Duality
Varying External Parameters

- Varying $B$ causes transitions from one plateau to the next.
- Varying $T$ removes the area between the plateaux
Temperature Dependence

- For those $B$ where $\sigma_{xy}$ lies on a plateau, $\rho_{xx}$ goes to zero at low temperatures.
  - $\sigma_{xy} = -\nu \frac{e^2}{h} ; \nu = \frac{-p}{q}$

- Hall Insulator: for very large $B$, $\rho_{xx}$ instead goes to infinity at low temperatures.

Yang et.al., cond-mat/9805341
Critical Points

- At plateaux, $\rho_{xx}$ falls as $T$ falls.
- For large $B$, $\rho_{xx}$ eventually rises as $T$ falls, (and so is an insulator).
- The critical field, $B_c$, for the plateau-insulator transition, occurs when $\rho_{xx}$ is constant as $T$ falls.

Yang et.al., *cond-mat/9805341*
Critical Resistivity is Universal

Yang et.al., cond-mat/9805341

- Positions of fixed points are independent of sample.
- e.g. For $\nu = 1/(2n+1)$ to Hall Insulator transitions:
  $$\rho_{xx}^c = \frac{h}{e^2} = 25.3 \text{ k}\Omega$$
Critical Scaling

- Near critical points predict universal form:
  - $\xi = \text{localization length of electrons}$
  - $\sigma_{xx} = f[\Delta B \xi^a]$

- Conductivity approaches its critical value as a universal power law as a function of:
  - temperature if $\xi$ varies as $T^p$
  - system size, if $\xi$ varies as $L$
  - $\sigma_{xx} = f[\Delta B / T^K]$
Superuniversality

- Slope of Hall resistivity between plateaux diverges like $1/T^b$
  - Same power of $T$ regardless of which plateaux are involved!
- Width of transition region vanishes like $T^b$
  - Same power of $T$ (up to sign) as for previous divergence

Wei et. al., *PRL* 61 1294.
B Dependence of Flow

Shahar et.al., cond-mat/9706045

For \( v=1 \) to Insulator transition:

\[
\rho_{xx} = \rho_{xx}^c \exp \left[ -\frac{\Delta v}{\nu_0(T)} \right]
\]
Deviations from Scaling?

Some disordered samples do not seem to have scaling form.

Temperature scaling should imply $\rho_{xx}$ is a function only of the combination: $\Delta\nu/T^K$.

Shahar et al., cond-mat/9706045
Flow in the Conductivity Plane

- As $T$ goes to zero (IR), $\sigma_{xx}$ flows to zero.
  - For plateaux $\sigma_{xy} \bullet 0$ so flow is to fractions on the real axis away from origin.
  - For Hall insulator $\sigma_{xy}$ also flows to zero, corresponding to flow to the origin.
- Critical behaviour occurs at the bifurcation points.
Semicircle Law

- Varying B for fixed T causes transitions between plateaux.
  - In clean systems flow is along a semicircle in the $\sigma$ plane.
  - In $\rho$ plane this corresponds to flow with $\rho_{xx}$ varying from 0 to infinity, and $\rho_{xy}$ fixed.
Semicircle Evidence I

Semicircle: An exact relation in the Integer and Fractional Quantum Hall Effect

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We present experimental results on the quantized Hall insulator in two dimensions. This insulator, with vanishing conductivities, is characterized by the quantization (within experimental accuracy) of the Hall resistance in units of the quantum unit of resistance, $\hbar/e^2$. The measurements were performed in a two dimensional hole system, confined in a Ge/SiGe quantum well, when the magnetic field is increased above the $\nu=1$ quantum Hall state. This quantization leads to a nearly perfect semi-circle relation for the diagonal and Hall conductivities. Similar results are obtained with a higher mobility n-type modulation doped GaAs/AlGaAs sample, when the magnetic field is increased above the $\nu=1/3$ fractional quantum Hall state.

PACS numbers: 73.40.Hn, 71.30.+h, 72.80.Sk

In the extreme quantum limit, when the magnetic field (B) exceeds a critical field $B_c$, the quantum Hall series is terminated by an insulator. This insulator is characterized by a diverging diagonal resistivity, $\rho_{xx}$, as the temperature (T) vanishes. In general this insulating behavior occurs when the lowest resolved energy level exceeds the Fermi energy. For samples exhibiting only integer plateaux the transition occurs typically beyond $\nu=1$. For higher mobility samples this transition can occur beyond the $\nu=1/3$ state or the $\nu=1/5$ state. In addition to these primary fractions transitions to insulating behavior have been observed originating from many other fractions. In this article we will only concentrate on the transitions from the states $\nu=1$ to insulator and $\nu=1/3$ to insulator. For both these...
Semicircle Evidence II
As $\rho_{xx}$ varies (for fixed $\rho_{xy}$) across the critical point, $\rho_{xx} = 1$, it goes into its inverse for opposite filling factors.

$$\rho_{xx} \left( -\Delta \nu \right) = \frac{1}{\rho_{xx} \left( \Delta \nu \right)}$$
An underlying symmetry: phenomenology
Symmetry and Superuniversality

- Superuniversality strongly suggests a symmetry:
  - Resulting symmetry group: $\Gamma_0(2)$ (plus particle-hole interchange)

- *Lütken & Ross*: postulate symmetry acting on complex $\sigma$.
  - Motivated by Cardy’s discovery of $\text{SL}(2,\mathbb{Z})$ invariant CFT.

- *Kivelson, Lee & Zhang* use Chern-Simons-Landau-Ginzburg theory to argue for symmetry acting on filling fractions, $\nu$

\[ T : \sigma \to \sigma + 1 \]
\[ ST^2S : \sigma^{-1} \to \sigma^{-1} - 2 \]
\[ \sigma \to 1 - \sigma^* \]
Symmetry Implies Flow Lines

C.B., R. Dib, B. Dolan, *cond-mat/9911476*
Flow Lines Imply QH Observations

B. Dolan, cond-mat/9805171; C.B., R. Dib, B. Dolan, cond-mat/9911476

- Flows along vertical lines and semicircles centred on real axis.
- Implies observed properties of transitions:
  - IR attractive fixed points are fractions $p/q$, with $q$ odd.
  - Semicircle Law
  - Transitions from $p_1/q_1$ to $p_2/q_2$ are allowed if and only if
    
    $$(p_1 q_2 - p_2 q_1) = \pm 1.$$  

- Universal Critical Points: $\sigma^* = (1+i)/2$
- Superuniversality of critical exponents
- Contains duality: $\rho_{xx}$ to $1/\rho_{xx}$
Quasi-microscopic symmetry: Particle/Vortex Duality

C.B., B. Dolan, *hep-th/0010246*
Microscopic Symmetry Generators

- Assume the EM response is governed by the motion of low-energy quasi-particles and/or vortices.

- Identify the three symmetry generators as:
  - \( ST^2S(\sigma) = \sigma/(1-2\sigma) \): Addition of 2\(\pi\) statistics;
  - \( TST^2S(\sigma) = (1-\sigma)/(1-2\sigma) \): Particle-vortex interchange;
  - \( P(\sigma) = 1 – \sigma^* \): Particle – hole interchange.

- Goal: calculate how EM response changes under these operations.
In 2 space dimensions, weakly-interacting particles and vortices resemble one another.

- EM response due to motion.
- Characterized by number, mass, charge/vorticity

One symmetry generator is particle-vortex duality

- System A:
  - $N$ particles of mass $m$
- Dual to A:
  - $N$ vortices of mass $m$
Low-Energy Dynamics

- Degrees of Freedom:
  - EM Probe: $A$
  - Statistics Field: $a$
- Systems with particles:
  - Particle positions: $x$
    - Charge, Mass
- Systems with vortices:
  - Vortex positions: $y$
    - Zeroes in order parameter for $U(1)_{\text{EM}}$ breaking
  - Goldstone Mode: $\phi$
    - Phase of order parameter
    - Mediates long-range Magnus force between vortices (unless eaten)

\[
L = \frac{\pi}{2\theta} \, a \wedge da + L_p(x, a + A) \\
L_p(x, a + A) = \frac{m}{2} \phi^2 - \phi(a + A)
\]

\[
\tilde{L} = \frac{\pi}{2\theta} \, a \wedge da + [\partial \phi - (a + A)]^2 \\
\tilde{L} = \frac{\pi}{2\theta} \, a \wedge da + b \wedge d(a + A) + L_v(y, b) \\
L_v(y, b) = \frac{m}{2} \phi^2 - 2\pi \, \phi \wedge b
\]
Relating EM Response for Duals

- Integrating out particle/vortex positions gives effective theory of EM response.
- Can relate dual EM responses without being able to evaluate path integrals explicitly.
  - Relies on equivalent form taken by $L_p$ and $L_v$.
- Ditto for $\theta \gamma - \theta + 2\pi$.

\[ e^{i\Gamma(A)} = \int [dx][da] e^{iS(x,a+A)} \]
\[ e^{i\Gamma(A)} = \int [dy][da][db] e^{i\tilde{S}(y,b,a+A)} \]
\[ \tilde{\Gamma}_{\theta=\pi}(A) = \Gamma_{\theta=-\pi}(A) + \frac{1}{2} \int dx \ A \wedge dA \]
Linear Response Functions

- For weak probes can restrict to quadratic power of $A_\mu$
- Response function $\Pi$ has convenient representation as a complex variables.
- Can relate response functions for dual systems.

$$\Gamma(A) = \frac{1}{2} \int dxdy \ A_\mu \Pi^{\mu\nu} A_\nu$$

$$\Pi^{\mu\nu} = \Pi_1(q^2) \Lambda^{\mu\nu} + \Pi_3(q^2) J^{\mu\nu}$$

$$\Pi = \Pi_1 + i\Pi_3$$

$$\frac{\Pi}{q^2} = \frac{i \sqrt{q^2} \left( \frac{\pi}{\theta} \right) - \Pi}{q^2 + i \sqrt{q^2} \left( \frac{\pi}{\theta} + \frac{\theta}{\pi} \right) \Pi}$$
EM Response and Duality

- Momentum dependence of $\Pi$ distinguishes EM response:
  - Superconductors
    - $A$ nonzero
  - Insulators
    - $A$ zero
  - Conductors
    - Square root branch cut
- Conductor $\diamond$ Conductor
- Superconductor $\diamond$ Insulator

\[ \Pi_1 = A + Bq^2 + \Lambda \]
\[ \Pi_3 = \sqrt{q^2} [\sigma_{xy} + \Lambda] \]
\[ \Pi_1 = \sqrt{q^2} [\sigma_{xx} + \Lambda] \]
\[ \Pi_3 = \sqrt{q^2} [\sigma_{xy} + \Lambda] \]
Duality for Conductors

- For conductors:
  - PV duality acts on conductivity plane.
  - Ditto for addition of $2\pi$ statistics.

- Duality not symmetry of Hamiltonian but of RG flow:

For conductors:
\[
\sigma = \left( \frac{\pi}{\theta} \right)^{-\sigma} \frac{1}{1 - \left( \frac{\pi}{\theta} + \frac{\theta}{\pi} \right) \sigma}
\]

\[
\sigma' = \frac{\sigma}{1 - 2\sigma}
\]
Duality Commutes with Flow

- Suppose a system with given conductivities corresponds to a system of $N$ quasiparticles, and its dual is $N$ vortices.
- Response to changes in $B, T$ computed by asking how $N$ objects of mass $m$ respond, and so is the same for system and its dual.
\( \rho_{xx} \) to \( 1/\rho_{xx} \) Duality

- Particle-vortex interchange corresponds to the group element which maps the circle from 0 to 1 to itself.
  - \( \rho_{xx} \downarrow \diamond 1/\rho_{xx} \)
- Particle/vortex interchange for duals implies filling factors are related by
  - \( \Delta v \downarrow \diamond -\Delta v \)

Shahar et al., *Science* 274 589
Duality for Bosons
Duality for Boson Charge Carriers

- Choose particles to be bosons:
  - Resulting symmetry group: $\Gamma_0(2)$ (plus particle-hole interchange)

- Predictions for the QHE can be carried over in whole cloth
  - Several precursor applications of this symmetry, or its subgroups
    - Lütken
    - Wilczek & Shapere
    - Rey & Zee
    - Fisher

\[
\begin{align*}
\sigma & \rightarrow -1/\sigma \\
\sigma^{-1} & \rightarrow \sigma^{-1} + 2 \\
\sigma & \rightarrow -\sigma^* 
\end{align*}
\]
\( \Gamma_\theta(2) \)-Invariant Flow

- Duality predicts flow lines, and flow lines predict effects to be expected:
  - Flow in IR to even integers and fractions \( p/q \), with \( pq = \text{even} \)
  - Semicircle Law as for fermions
  - Rule for which transitions are allowed.
  - Universal Critical Points:
    - \( \sigma_* = i \) (metal/insulator transition?)
  - Superuniversality at the critical points
  - Duality: \( \sigma_{xx} \) to \( 1/\sigma_{xx} \) when \( \sigma_{xy} = 0 \)
$\Gamma_\theta(2)$-Invariant Flow
The Bottom Line

- Particle-Vortex Duality relates the EM response of dual systems
  - Implies symmetry of \textit{flow}, not of Hamiltonian
  - Combined with statistics shift generates large discrete group
- Works well for QHE when applied to fermions
  - Many predictions (semicircles \textit{etc}) follow robustly from $\Gamma_0(2)$ symmetry implied for the flow in QH systems.
- Implies many new predictions for QH systems when applied to bosons.
  - Eventually testable with superconducting films, Josephson Junction arrays, etc.
  - Potentially many more implications…
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