MULTIFRACTALS AND EXTREME RAINFALL EVENTS

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Abstract. Based on a multifractal structure hypothesis for temporal rainfall processes, a general formula relating maximum possible point rainfall accumulations is derived as a function of the duration and sample size. This formula appears to be in agreement with empirical observations. Such a result may reconcile some opposite points of view regarding extreme rainfall events, and suggests new ways of exploiting the scaling properties of rain processes.

From fractal sets to multifractal fields:

Fractals were first seen from a purely geometrical point of view [Mandelbrot, 1975, 1982]. Applied to rainfall time series studies this approach only enabled us to cope with occurrence or non-occurrence [Hubert and Carbonnel, 1989] of rainfall or monofractal functions whose fractal dimension was independent of the intensity threshold [Lovejoy, 1981; Lovejoy and Mandelbrot, 1985; Lovejoy and Schertzer, 1985]. The occurrences must be carefully defined according to some kind of rainfall intensity threshold. The observed dependence of the fractal dimension upon the value of this threshold then yields an infinite hierarchy of dimensions [Schertzer and Lovejoy, 1985, 1991; Lovejoy et al, 1987; Lovejoy and Schertzer, 1990]. At first sight, we have lost the simplicity of a unique fractal descriptor, but this problem can be overcome by using (universal) multifractal fields rather than fractal sets. The basic equation of multifractal fields, derived from the theory of multiplicative cascades, is the following [Schertzer and Lovejoy, 1987a,b, 1989]:

\[ Pr(R_\lambda > \lambda^\gamma) = \lambda^{-c(\gamma)} \] (1)

Where \( \lambda \) is the scale ratio and is \( T/\tau \) in time series of length \( T \) divided in elementary time-periods \( \tau \), \( R_\lambda \) is the intensity of the field at scale ratio \( \lambda \), \( \gamma \) is an order of singularity. This equation states how the probability distribution of singularities of order higher than a value \( \gamma \) is related to the fraction of the space they occupy as determined by the codimension \( c(\gamma) \). The "\( = \)" sign in the right side of equation (1) indicates equality to within slowly varying (e.g. log) functions of \( \lambda \). For some results on multifractals and rain (including a review) see Schertzer and Lovejoy (1987a, 1989), Lovejoy and Schertzer, (1992).

Scaling laws for extreme rainfalls

We may now consider extreme rainfall events occurring within a duration \( \tau \). Whenever there is a maximum order of singularity \( \gamma_{max} \), then the maximum accumulation \( A_\lambda \) will be:

\[ A_\lambda = \lambda^{-1} R_\lambda T \propto \lambda^{\gamma_{max}-1} \propto \tau^{1-\gamma_{max}} \] (2)

As predicted by eq. 2, the maximum recorded point rainfall depths for different durations (from minutes to several years), as presented in most hydrological or meteorological textbooks...
and encyclopedia [Gilman, 1964; Rémeniéras, 1965; Raudkivi, 1979] exhibit more or less the same algebraic behavior (figure 1). On a log-log diagram, the slope is found to be about 0.5 ($\gamma_{\text{max}} = 0.5$).

There are many mechanisms which can give rise to finite $\gamma_{\text{max}}$. In multifractal processes, the highest possible order of singularities can be artificially restricted by construction (e.g. "geometric" or "microcanonical" processes [Schertzer and Lovejoy, 1992]). However, such a maximum is naturally produced by a large class of universal processes [Schertzer Lovejoy, 1987a,b, 1991; Schertzer et al, 1991; Brax et Pechanski, 1991; Schmitt et al, 1992]. In this paper, we don't discuss the various issues related to universality in multifractals, i.e. existence of attractive and stable processes which are scaling nonlinear analogues of the gaussian noises that often arise in linear processes. We rather concentrate on the exploitation of their codimension function below which depends only on two fundamental parameters $C_1$ and $\alpha$ (for non conservative process, a third parameter $H$ is necessary):

$$c(\gamma) = C_1 \left( \frac{\gamma}{C_1 \alpha} + \frac{1}{\alpha} \right)^{\alpha-1} \quad \alpha \neq 1$$

$$c(\gamma) = C_1 \exp \left( \frac{\gamma}{C_1 - 1} \right) \quad \alpha = 1$$

(3)

where $\alpha^{-1} + \alpha^{-1} = 1$. $C_1$ measures the departure from homogeneity ($C_1 = 0$) and $\alpha$ is the Lévy index bounded between 0 and 2, which measures the departure from monofractality ($\alpha = 0$).

For $\alpha \geq 1$, the orders of singularity are unbounded, on the contrary when $0 < \alpha < 1$, a finite maximum order of singularity $\gamma_0$ does exist:

$$\gamma_0 = C_1$$

(4)

Sampling dimension to quantify limited information

We must now consider a complication which arises because of the finite size of the data sets. A finite sample will miss extreme events which are sufficiently rare: the empirical singularities will be bounded by an effective maximum $\gamma_s$ smaller than the theoretical $\gamma_0$. Indeed, if we have $N_s$ independent series, each with range of scales $\lambda$, then we may introduce the "sampling dimension" [Schertzer and Lovejoy, 1989, 1991; Lavallée, 1991; Lavallée et al., 1991] ($D_s$) to quantify the fraction of the (infinite) dimensional probability space actually explored:

$$D_s = \log N_s \quad \log \lambda$$

The actual total dimension is $D + D_s$, where $D$ is the dimension of each sample, $D=1$ for time series. We now use the geometric interpretation of $c(\gamma)$ as a codimension: whenever the observing space $D + D_s > c(\gamma)$, the set of singularities of order $\gamma$ will be a fractal set of dimension $D + D_s - c(\gamma)$. The maximum attainable $\gamma$ yielding a nonnegative dimension (see figure 2), is $\gamma_s = C_1^{-1} (D + D_s) = \gamma_0 [1 - (C_1/(D + D_s))^{-1/\alpha}]$ which satisfies (for $\alpha < 1$, $\alpha < 0$):

$$\gamma_0 \left( 1 - \alpha \left( \frac{C_1}{D} \right)^{-\frac{1}{\alpha}} \right) \leq \lambda_s \leq \gamma_0$$

(6)

with upper bound corresponding to an infinite sample size ($D_s \rightarrow \infty$), the lower bound to a single sample ($D_s = 0$).

Confrontation with empirical observations

Glancing at the empirical values gathered in table 1, it is noteworthy that $\alpha$ is quite consistently in the vicinity of 0.5 and certainly bounded by 0 and 1. As a consequence, in the framework of universal multifractals, rainfall time series should have a maximum order of singularities. Averaging

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Fig. 1: The world's record point rainfall values, reproduced from Raudkivi (1979). 1 - Cherrapunji, India; 2 - Silver Hill Plantation, Jamaica; 3 - Funkiko, Taiwan; 4 - Baguio, Philippine Is.; 5 - Thrall, Texas; 6 - Smethport, Pa; 7 - D'Hani, Texas; 8 - Rockport, W.Va; 9 - Holt, Mo.; 10 - CCL de Arges, Romania; 11 - Plumb Point, Jamaica; 12 - Fussen, Bavaria; 13 - Unionville, Md.; values from Jennings Jenning, (1950). (+) La Reunion, France; (o) Paishih, Taiwan; values from Paulhus (1965).
Fig. 2: The $c(\gamma)$ curve corresponding to the estimated parameters $\alpha = 0.51 \pm 0.05$, $C_1 = 0.44 \pm 0.16$, with $\gamma_s$ for $N_s = \lambda D_s$ samples ($N_s = \lambda D_s$, $\lambda$ being the scale ratio and $D_s$ the sampling dimension), and $\gamma_0$ ($= \gamma_s$ for an infinite number of samples; $D_s = N_s = \infty$).

over all the analyzed series we obtain $\alpha = 0.51 \pm 0.05$, $C_1 = 0.44 \pm 0.16$. In the following we will assume that the embedding space dimension $D$ is equal to 1 and that the sampling dimension $D_s$ is equal to zero (i.e. considering a unique sample). Due to the nonlinearity of eqs. 4 and 6, there are two direct ways of obtaining the statistically best estimate of $\gamma_0$ and $\gamma_s$, yielding different results. On the one hand, as displayed in table 1, we can compute estimates of $\gamma_0$ and $\gamma_s$ from each estimate of $\alpha$ and $C_1$, then average. On the other hand, we can compute estimates of $\gamma_0$ and $\gamma_s$ from averaged values of $\alpha$ and $C_1$. However, it turns out that the differences between the two results are not so important: $\gamma_0 = 0.88$ and $\gamma_s = 0.66$ with the first method; $\gamma_0 = 0.90$ and $\gamma_s = 0.69$ with the second method. We actually prefer the first method because it readily yields a confidence interval: $\gamma_0 = 0.88 \pm 0.31$; $\gamma_s = 0.66 \pm 0.20$.

The observed slope of figure 1 ($\approx 0.5$), is compatible within one standard deviation with that estimated from $\gamma_s$ values ($1 - \gamma_s = 0.34 \pm 0.20$) and quite compatible within this deviation from that estimated from $\gamma_0$ values. More refined theories could take into account a possible degree of non conservation of the rain cascade, as well as the observed breaks in the scaling. Other improvements could include new estimation algorithms as well as the processing of larger data sets and may lead to even closer theoretical and empirical agreement.

Conclusions and perspectives

There exist presently two rather opposite views on extreme precipitation. One school of thought relies deeply on the notion of the "possible maximum precipitation" (PMP) considered as a physically based notion. In order to estimate the possible maximum precipitation at a given location (and implicitly at a given scale) a sophisticated analysis of the rainfall process in an attempt to address all its relevant and physical aspects (meteorology, orography, etc.) is required. However, such an approach is often considered as remaining too speculative or qualitative, especially with respect to engineering needs.

On the other hand, supporters of statistical analysis consider rainfall rate as a random variable and time series as a stochastic process. Statistical approaches lead to rainfall rate probabilities useful in engineering designs. However, without any reference to any physical processes, the role of hydrologists could easily be reduced to fitting empirical data to ad hoc statistical laws.

These early results may help to reconcile the two points of view since they are based on both physics and statistics. Indeed, in our approach the multiplicative cascade accounts for turbulent processes resulting from nonlinear interactions between different scales and fields and leads to the statistical

Table 1: A comparison of various gage estimates of $\alpha$, $C_1$, $\gamma_0$, $\gamma_s$ over various time scales. Parameters were mostly estimated from PDMS (Probability Distribution/Multiple Scaling) the DTM (Double Trace Moment) techniques [Lavallée, 1991; Lavallée et al, 1991; Lovejoy and Schertzer, 1991; Schertzer and Lovejoy, 1991]. (*Private communications; +Ladoy et al., submitted to C. R. Acad. Sci. Paris, 1992).

<table>
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<th>Data Type</th>
<th>Gage, daily accumulation</th>
<th>Gage, 6 minutes resolution</th>
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<td></td>
<td>(scaling regime up to 16 days)</td>
<td>(scaling regime up to 30 days)</td>
<td>(scaling regime up to 16 days)</td>
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<tr>
<td>$\alpha$</td>
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<td>0.5</td>
<td>0.45</td>
<td>0.59</td>
<td>0.50</td>
<td>0.51 \pm 0.05</td>
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<td>$C_1$</td>
<td>0.6</td>
<td>0.20</td>
<td>0.6</td>
<td>0.32</td>
<td>0.47</td>
<td>0.44 \pm 0.16</td>
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<tr>
<td>$\gamma_0$</td>
<td>1.20</td>
<td>0.40</td>
<td>1.09</td>
<td>0.78</td>
<td>0.94</td>
<td>0.88 \pm 0.30</td>
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<tr>
<td>$\gamma_s$</td>
<td>0.84</td>
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<td>0.83</td>
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<td>0.72</td>
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description of rainfall (eqs.1, 3). We are thus able to give a precise (statistical) definition of the possible maximum precipitation at a given scale: we not only clarify the role of scales for the definition of the PMP, but also the role of the limited size of samples used for its estimation. We furthermore showed that the two basic multifractal exponents \( C_1, \alpha \) determine the maximum attainable singularities \( \gamma_0 \) and \( \gamma_\alpha \) and hence the possible maximum precipitation at a given scale and on a given sample.

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References


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