Fractal properties of rain, and a fractal model

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ABSTRACT

This paper amplifies upon earlier theories and observations by the authors, advances a probabilistic model of rain fields, and exhibits realistic simulations, which the reader is advised to scan before continuing with the text. The model is also compared with other approaches. The theory of fractals has been in part motivated by the Hurst effect, which is an empirical observation in hydrology and climatology. A fractal is an unsmooth shape that is scaling, that is, where shape appears unchanged when examined by varying magnifications. The study of fractals is characterized by the prevalence of hyperbolically distributed random variables, for which \( \text{Pr}(U > u) \propto u^{-H} \), where \( \text{Pr} \) is the probability that the value of the variable exceeds \( u \), and \( H \) is a positive exponent. Lovejoy established the applicability of fractals in meteorology, by showing that cloud and rain areas project on the Earth along shapes whose boundaries are fractal curves, and that the temporal and spatial structure of rain is rife with hyperbolically distributed features. These observations, as amplified in this paper, set up the challenge of constructing fractal models with the observed properties. The models presented here belong to a very versatile family of random processes devised in Mandelbrot: fractal sums of (simple) pulses, or FSP processes. The simulations to be presented reveal that these processes involve a scaling hierarchy of "bands", "fronts", and "clusters"; as well as other complex shapes, none of which had been deliberately incorporated in the process. This very rich morphology and the related statistical distributions exemplify the power of simple fractal models to generate complex structures, and are in accord with the wide diversity of actual rainfall shapes. It is argued that this model already provides a useful context in which the basic statistical properties of the rain field, including the relationship between the temporal and spatial structure, may be studied.

1. Introduction

A striking feature of rainfall is its extreme spatial and temporal variability. Over a wide range of time and space scales, sudden changes occur so frequently that rainfall is often intuitively described as "erratic". In fluid mechanics, the analogous phenomenon of abrupt transitions is called "intermittency", which is the terminology we use here. Mandelbrot’s systematic studies of intermittency have revealed its profound links with scaling and with hyperbolic distributions.

A phenomenon is called scaling when it involves no characteristic scale of length or time. In particular (Mandelbrot and Van Ness, 1968), a random function \( X(t) \) is called scaling at the origin, if \( X(0) = 0 \) and \( X(\Delta t) \overset{d}{=} \lambda^{H} X(t) \) for all ratios \( \lambda > 0 \).

Here \( \overset{d}{=} \) denotes identity in statistical distribution and \( H \) is a scaling exponent.

More generally, \( X(t) \) is scaling for all times if, for arbitrary \( t_{0} \) and \( t_{1} \), the symbolic relation \( \Delta X(\lambda \Delta t) \overset{d}{=} \lambda^{H} \Delta X(\Delta t) \) holds between the quantities \( \Delta t = t_{1} - t_{0}, \Delta X = X(t_{1}) - X(t_{0}), \Delta t = t_{1} - t_{0} + \lambda(t_{1} - t_{0}) \) and \( \Delta X(\lambda \Delta t) = X(t_{1}) - X(t_{0}) \). When \( \lambda \) is large, this is a relation between large-scale variation over the long time lag \( \lambda \Delta t \), and small scale variation over the short time lag \( \Delta t \). The notation \( \Delta t, \Delta X \) is meant to emphasize that scaling refers to differences. The graphs of scaling functions, and other related geometric shapes are "fractal sets" (Mandelbrot 1975, 1977, 1982) characterized by a "fractal dimension" that is a non-integer quantity related to \( H \).

A random variable \( U \) is called hyperbolic if the
tail probability \( Pr(U > u) \) takes the form \( Pr(U > u) \propto u^{-\alpha} \). (The custom is to denote a random variable by a capital letter, and its value by the corresponding lower case letter.) Mandelbrot (1974, 1984a) showed that a broad family of intermittent fluctuations, now called “multiplicative chaos”, leads to such a distribution for the intermittent quantity. On the role of hyperbolic distributions in meteorology, see also Schertzer and Lovejoy (1984b).

The smaller the value of \( \alpha \), the greater are the extreme values of \( U \); in fact, the moment \( \langle U^h \rangle \) is finite for \( h < \alpha \) but diverges for \( h > \alpha \). This last property requires comment. Since sample moments are always finite, divergent theoretical moments used to be viewed as strange and abnormal, and as sufficient to make hyperbolic random variables “improper” for modeling nature. In fact, a divergent theoretical moment merely expresses that the sample moment does not converge to any limit and can become arbitrarily large. Mandelbrot (1963a, b) was the first to show that such is indeed the case in important natural fluctuations: not only do hyperbolic random variables fit a broad variety of data, but samples constructed with such variables turn out to contain a variety of “configurations” that often seem so clear-cut as to make it difficult after the fact to believe that they are due to “mere chance”. On a broader scene (Mandelbrot, 1975, 1977, 1982), very simple fractal mechanisms prove to have the ability to generate seemingly complex samples. In any event, the hyperbolic distribution’s properties differ sharply from those of the customary Gaussian distribution and from those of its elementary transforms such as the lognormal distribution.

Lovejoy tested the applicability of scaling, fractals, and of the hyperbolic distributions to the study of the rainfall. In particular, Lovejoy (1982) established the absence of fundamental length scales and hence the fractal nature of the perimeters of rain and cloud fields (at least from \( 1 \text{ km}^2 \) to \( 1.2 \times 10^8 \text{ km}^2 \)), and Lovejoy (1981) established the existence of hyperbolic fluctuations in rain rate from many areas of the world. One purpose of this paper is to record these experimental results in more complete and more accessible form.

This paper’s second purpose is to study stochastic fractal models which produce and hyperbolically intermittent fields, a family of fractals, introduced in (1984b), involves fractal sums of simple specific models used here involves only \( \alpha = 3 \) and seems rather structureless. Yet a great many simulations reported later in this paper, they are true to what one should expect and reproduce many of the character (Fig. 1) to be present in rainfields: (i) intermittency, (ii) complexity of form, (iii) hierarchy of line-like structures, (iv) theory of rain areas, (v) clustering of “cells” and so on. Section 2 examines the empirical evidence for scaling and hyperbolic intermittency. Section 3 describes a scaling stochastic model. Section 4 applies this model to various rainfall statistics. Section 5 compares this scaling model to earlier rainfall statistics, Section 6 to time and space lags, and Section 7 to general arguments in favor of scaling and fractals. Finally, Section 6 is an empirical study.

2. Empirical evidence for scaling and hyperbolic intermittency in rainfall

2.1 Evidence for hyperbolic intermittency

It is natural to observe the fluctuation of rainfall from an isolated rain area. This was done with radar data in Montreal, Spain, and on the North Atlantic with \( 4 \times 4 \text{ km} \), and 5 m resolution. By their nature, rain areas can split apart or merge together; it is necessary to establish a criterion for the former and selection.

In a first study, reported in Fig. 2a, I picked those areas that had maintained identity for at least 30 minutes with at least \( 1 \text{ cm}^3 \) for 1.2 \( \times 10^6 \text{ km}^2 \), and Lovejoy (1981) established the existence of hyperbolic fluctuations in rain rate from many areas of the world. One purpose of this paper is to record these experimental results in more complete and more accessible form.

![Fig. 1: Radar rain map at 3 km altitude taken by H.M.S. Quadra in the tropical Atlantic. Symbols, first and then in alphabetic order, indicate rain rates on a log scale. 8 symbols per decade in rain rate. Resolution 5 km. Asterisks indicate sea clutter noise (center only), and the maximum radar range (250 km). Note the straight lines, commonly referred to as “bands” or “fronts.”](image)
FRACTAL PROPERTIES OF RAIN, AND A FRACTAL MODEL

2. Empirical evidence for scaling and hyperbolic intermittency in the rain field

2.1. Evidence for hyperbolic temporal intermittency

It is natural to observe the fluctuations in the flux of rain from an isolated rain area. This was done with radar data in Montreal, Spain and the tropical Atlantic with 4 x 4 km spatial, and 5 min temporal resolution. By their nature, rain areas frequently split apart or merge together; it is therefore necessary to establish a criterion for their definition and selection.

In a first study, reported in Fig. 2a, b, c, Lovejoy picked those areas that had maintained their identity for at least 100 min, with only minor splits. Rain areas were followed until a major split or merger occurred, after which the area was considered “dead” (see Tsonis and Austin (1981) for details). Less than 10% of all areas satisfied these criteria. Minor splitting is allowed, but...
Fig. 2. The probability \( \Pr(\Delta R > \Delta r) \) that the absolute value of a negative change in rain flux \( |\Delta r| \) exceeds a given \( \Delta r \). The intervals between measurements are as follows: white circles: 5 min; black circles: 10 min; white triangles: 20 min; black triangles: 40 min. Straight lines have slopes 1.65. Data from (a) two afternoons in Spain, and 21 different rain areas selected as described in the text, (b) 29 storms in the tropical Atlantic (GATE experiment) over 4 different days, and (c) 7 storms in Montreal over 1 afternoon.

Fig. 3. The probability \( \Pr(\Delta R > \Delta r) \) of the absolute change in rain flux over 5 min, \( |\Delta r| \) exceeding a fixed \( \Delta R \). Positive changes, black circles; negative changes, white circles. Data are from the tropical Atlantic; rain areas were selected as described in the text.

Telus 37A (1985), 3
the tropical Atlantic are normalized so as to yield the same mean flux, the distributions of AR cannot be statistically distinguished by Komolgorov–Smirnov tests. If the positive and negative Δr's are normalized separately, they are also distributed similarly.

Additional support for hyperbolic behavior with α ~ ½ comes from Fig. 3, where a different rain area selection criterion was used. All areas that neither split nor merged on two consecutive scans were used—about 50% of the areas present. During the 3 h during which data were collected, the total flux of all storms increased by 40%. However, positive and negative Δr's were almost equal in number (238 and 213 respectively). Most of the increase is therefore due to the fact that in absolute value, the large positive Δr's exceed by ~50% the large negative Δr's (Fig. 3). Subintervals exhibiting R and decreasing flux yielded similar distributions, except that the relative magnitude of the distributions of large positive and large negative Δr's was different. In all cases, the (Δr)^{1.65} asymptotic behavior was unaffected.

It should be noted that these results are unlikely to be artifacts of the radar data processing. Indeed (Mandelbrot, 1963a), the hyperbolic tail of the distribution is in general insensitive to the presence of both additive and multiplicative types of noise. In the case of rainfall, these may be the radar reflectivity fluctuations due to incoherent scattering. (The only exception, which is not of concern here, is when the noise is itself hyperbolic with a lower α value). This insensitivity is supported by the agreement found here in the value of α for different radars in different parts of the world.

2.2. Evidence for temporal scaling

If we write the distributions shown in Fig. 2 as:

\[ \text{Pr}(\Delta R > \Delta r) \sim (\Delta r/\Delta r^*)^{-\alpha} \]

the quantity Δr* measures the “width” or amplitude of the ΔR. If the rainfall field is scaling in time, then Δr* ~ (Δt)^H (naturally, the coefficient of proportionality depends on the units of r and t). The subscript “L” is used here to indicate that rL is a “Lagrangian” parameter, relative to fluctuations following the flow. Doubling Δr implies (as observed in Fig. 2), that the “width” increases by the constant distance H log 2 on a log-log plot. The parameter H was estimated in two ways. (A) Sampling the intercurve distance in Fig. 2 four times every decade (starting for Pr < 0.1) yielded HL = 0.69 ± 0.06. (B) The best straight-line asymptote was fitted on the assumption that α = ½, thus determining Δr*(Δr), and HL was determined by a regression of log (Δr*) against log Δr. This yielded HL = 0.59. We conclude that HL = 0.64 ± 0.05.

In nature, scaling rules tend to hold over a finite scale range between a positive “inner cutoff” and a finite “outer cutoff”. The value of the outer cutoff is a very important characteristic of a problem. Very long time scales, which are usually associated with the climate, have been studied by Hurst (1951), who discovered a remarkable empirical rule that applies for example to river discharges and rain data (as well as to other geophysical variables). Mandelbrot (1965) argued that the puzzling “Hurst phenomenon” should be viewed as a symptom of scaling. Mandelbrot and Wallis (1968, 1969) broadened the scope of this explanation. Very short time scales, which are usually associated with turbulence, are also known to be scaling down to a
very short inner cutoff ruled by viscosity. A coherent picture of the entire spectrum from seconds to hundreds of thousands of years (similar to the study of temperature in Lovejoy and Schertzer (1983), see Subsection 4.4) would be illuminating.

2.3. Evidence for horizontal scaling

Using individual radar maps, we may evaluate $\Pr(\Delta R > \Delta r)$ for different spatial separations $\Delta r$. Two examples taken from GATE data are shown in Fig. 4. Note that they only differ in the width $\Delta r$. This was also found to be true in all other cases analyzed, namely in a total of 23 randomly chosen radar PPIs. From the separation between the curves, we estimate $H_e \sim 0.50$. The subscript "E" is used here to indicate that $H_e$ is relative to a "Euclidian" reference frame fixed relative to the flow. When using radar data to evaluate these distributions, care must be taken not to introduce any range-dependent radar data processing effects. Here, $\Delta R$ was determined cross-range in a narrow band, so as to compare equal-sized sampling volumes.

Note that in the diagram in Fig. 4, the asymptote is curved, meaning that the distribution is not hyperbolic, contrary to Figs. 2 and 3. We think this feature is associated with a certain degree of spatial smoothing of the extreme rainfall gradients. This effect is relatively easy to model and is discussed in Section 3. We note that the mean square $(\Delta r)^2$ is finite when taken in space, while it is infinite when taken in time because $\alpha = \frac{1}{2}$.

It is possible to estimate $H_e$ by a statistical method called $R/S$ analysis. The letters $R/S$ denote a quantity called "rescaled range", which was originally introduced by Hurst (1951) and subsequently used by Mandelbrot and Wallis (1969) and Mandelbrot (1972) as the basis for a new procedure for studying long-run dependence in empirical time series. This statistic is robust in the sense that its behavior is only influenced by long-run dependence, and is insensitive to large fluctuations around the long-term trend. $R/S$ analysis has the disadvantage that it is biased for short series of data. We applied it separately for data at ranges of 50, 100, 150 km from the radar. Applying $R/S$ analysis cross-range for separations of 20° to 50° (radar resolution of 1°) showed that $H_e = 0.48 \pm 0.02$ for $5 < \Delta X < 13$ km, $10 < \Delta X < 26$ km, $15 < \Delta X < 39$ km.

Either the probability distribution method or $R/S$ analysis may be used to estimate $H_e$, but both are limited in the scales over which they may exhibit scaling because of the limited range of radar data, and because of the necessity of avoiding range-dependent biasing effects with the radar.

The true limit of horizontal scaling is a basic meteorological problem, since all meteorological fields are strongly interrelated and the existence of a length scale in one may be expected to be reflected in the others. Furthermore, in the last 10 years, many experiments have attempted to detect length scales—particularly in the velocity field. These experiments have found no clear evidence of any basic length scale between the turbulence dissipation scale, of the order of centimeters, and planetary scales. Readers are referred to Schertzer and Lovejoy (1984a, b) for a review of the relevant empirical and theoretical arguments in favor of the scaling hypothesis.

Direct support for the scaling of the rain and cloud fields is found in Lovejoy (1982). Here, the fractal relation between the area $A$ and the perimeter $P$, as derived in Mandelbrot (1977, 1982), is used to show that $P \propto \sqrt{A}$ with $D = 1.35$, for rain and cloud areas between $1 \text{ km}^2$ and $1.2 \times 10^4 \text{ km}^2$. The coefficient of proportionality necessarily depends on the unit of length. This result has recently been extended down to $2.6 \times 10^{-2}$ km$^2$ with LANDSAT imagery (private communication, Cahalan et al. 1984). The exponent $D$ is the fractal dimension of the perimeter and is a measure of the perimeter's "complexity". Smooth perimeters are well known to satisfy $P \propto \sqrt{A}$; hence $D = 1$, and maximally complicated perimeters that literally fill the plane satisfy $P \propto A$, and hence $D = 2$. The fact that no deviations from $D = 1.35$ are found for lengths in the region from 0.1 km to 1000 km is direct evidence for scaling.

It would have been tempting to use the fractal perimeter relation to estimate $H_e$ since Mandelbrot (1977, 1982) shows that Gaussian distributions satisfy $H = D = 2$. However, the empirical value of this sum is $H_e + D = 1.83 < 2$. The discrepancy is not completely understood, but may indicate that the rain fields are non-Gaussian.

3. Description of the fractal rainfall

3.1 Mandelbrot's stable-Lévy model of the fractal

Before turning to our own, two-dimensional FSP models, it is convenient to translate in terms of rainfall time series the simple models of Mandelbrot (1963), which are not applicable to rain areas but demonstrate the basic features of the approach exploited in the more realistic FSP model. Mandelbrot's model shows the result of adding independent, identically distributed hyperbolic random variables, each having a density function $f(x)$ and a number of variables added is fixed, with the number of variables added is fixed and varies from 1 to 1300. Positive spacings were chosen with equal probability, and the sum is interpreted as a model of total rain flux from an isolated storm. Mandelbrot (1971) also shows that the sum of independently distributed hyperbolic random variables converges to a stable-Lévy variate with probability density is complicated asymptomatically hyperbolic, with an exponent $a$. Therefore, if the rule $P \propto \Delta r$ ($\Delta r$) is extended down to $\Delta r \rightarrow 0$, it would be useful to use the fractal perimeter relation to estimate $H_e$, since Mandelbrot (1977, 1982) shows that Gaussian distributions satisfy $H = D = 2$. However, the empirical value of this sum is $H_e + D = 1.83 < 2$. The discrepancy is not completely understood, but may indicate that the rain fields are non-Gaussian.
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\[ H_e = 0.48 \pm 0.02 \text{ for } 5 \text{ km } < \Delta X < 15 \text{ km}, 10 \text{ km } < \Delta X < 26 \text{ km}, 15 \text{ km } < \Delta X < 39 \text{ km}. \]

Either the probability distribution method or R/S analysis may be used to estimate } \eta \text{, but both are limited in the scales over which they may exhibit scaling because of the limited range of radar data, and because of the necessity of avoiding range-dependent biasing effects with the radar.

The true limit of horizontal scaling is a basic meteorological problem, since all meteorological fields are strongly interrelated and the existence of a length scale in one may be expected to be reflected in the others. Furthermore, in the last 10 years, many experiments have attempted to detect length scales—particularly in the velocity field. These experiments have found no clear evidence of any basic length scale between the turbulence dissipation scale, of the order of centimeters, and planetary scales. Readers are referred to Schertzer and Lovejoy (1984a, b) for a review of the relevant empirical and theoretical arguments in favor of the scaling hypothesis.

Direct support for the scaling of the rain and cloud fields is found in Lovejoy (1982). Here, the fractal relation between the area } A \text{ and the perimeter } P \text{, as derived in Mandelbrot (1977, 1982), is used to show that } P \propto \sqrt{A} \text{ with } D \sim 1.35 \text{, for rain and cloud areas between } 1 \text{ km}^2 \text{ and } 1.2 \times 10^6 \text{ km}^2. \text{ The coefficient of proportionality necessarily depends on the unit of length. This result has recently been extended down to } 2.6 \times 10^{-2} \text{ km}^2 \text{ with LANDSAT imagery (private communication, Cahalan et al. (1984)). The exponent } D \text{ is the fractal dimension of the perimeter and is a measure of the perimeter's "complexity". Smooth perimeters are well-known to satisfy } P \propto \sqrt{A}; \text{ hence } D \sim 1, \text{ and maximally complicated perimeters that literally fill the plane satisfy } P \propto A, \text{ and hence } D \sim 2. \text{ The fact that no deviations from } D \sim 1.35 \text{ are found for lengths in the region from } 0.16 \text{ km to } 1000 \text{ km is direct evidence for scaling.}

It would have been tempting to use the fractal area-perimeter relation to estimate } H_e, \text{ since Mandelbrot (1977, 1982) shows that Gaussian fluctuations satisfy } H + D = 2. \text{ However, the empirical value of this sum is } H_e + D \sim 1.85 < 2. \text{ This discrepancy is not completely understood, but may indicate that the rain field is definitely non-Gaussian.}

3. Description of the fractal rain model

3.1 Mandelbrot's stable-Levy model of the Noah effect, and storm time series

Before turning to our one-, two- and three-dimensional FSP models, it is instructive to translate in terms of rainfall time series a very simple model that has proven useful in economics and elsewhere (Mandelbrot 1963b, 1982). This model is not applicable to rain areas, but it already demonstrates the basic features of intermittency exploited in the more realistic FSP models. Fig. 5 shows the result of adding independent identically distributed hyperbolic random variables with } \alpha = \frac{1}{2}; \text{ the number of variables added is labeled as time and varies from 1 to 1300. Positive and negative increments were chosen with equal probability, and the sum is interpreted as a model of the time series of total rain flux from an isolated storm. For } \alpha < 2, \text{ a generalized central limit theorem (see e.g. Feller (1971)) shows that the sum of independent identically distributed hyperbolic random variables converges to a stable-Levy variable. This limit's probability density is complicated, being only asymptotically hyperbolic, but with the same exponent } \alpha. \text{ Therefore, if the rule } \Pr(\Delta R > \Delta r) \propto (\Delta r)^{-\alpha} \text{ is extended down to } \Delta r \to 0, \text{ the result is}
scaling. In particular, independent increments yield $H = 1/\alpha - \frac{1}{2} = 0.60$. This model therefore has stable-Levy increments. Fig. 5 is drawn with values of $\alpha$ and $H$ close to those of actual rainfall data analyzed in Figs. 2 and 3. Since successive fluctuations are independent in Fig. 5, the best prediction concerning future values of $R(t)$ is that there will be no persistence, that is, no change on the average. This conclusion agrees with the results of Tsions and Austin (1981) and Tsions et al. (1984), but may not hold for all processes with similar $\alpha$ and $H$. It would be interesting to examine the question of predictability in the model presented below.

Fig. 5 illustrates a basic property of random functions whose increments are hyperbolically distributed with $\alpha < 2$. A priori, these increments are identically distributed. A posteriori, however, the largest increment is most unusual: it is of the same order as the sum of all the others, and thus dominates this sum. In Fig. 5, this effect may be seen by inspecting any interval: a large fraction of the total change in $R(t)$ comes from a single large "jump". Mandelbrot and Wallis (1968) called this feature the "Noah effect" after the extreme 40-day, 40-night fluctuation responsible for the biblical flood. The fact that a single large fluctuation may dominate the others is quite different from the situation made familiar by the Gaussian or quasi-Gaussian cases, where individual fluctuations rarely exceed several standard deviations, and even the largest fluctuation is negligible compared to the sum — this being a criterion for the sum's convergence to the Gaussian. We show below that the counterparts of this fact in more refined models permit a much greater richness in structure, including a scaling hierarchy of clusters and line-like shapes.

3.2. Characteristic scales of the rectangular pulses (RSP) in one dimension

Consider (Fig. 6) a function $R(t)$ that is the sum of rectangular pulses of random heights, representing a rainfall intensity increment $\Delta R$, random widths, representing the rainfall duration $p$, and random centers, distributed according to a Poisson process with a constant rate $n$. In the framework of the usual stochastic models, one assumes (without saying so) that the expectations relative to the pulses, $\langle p \rangle$ and $\langle \Delta R \rangle$, are both finite and are taken to constitute two characteristic scales. An implementation of these assumptions is found in Rodriguez-Iiturbe (1983). Under these assumptions, however, the sum of the pulses is a strongly scale-dependent process, whose properties for $T = \langle p \rangle$ are completely different from its properties for $T = \langle \Delta R \rangle$.

Mandelbrot (1984c) advances several ideas of insuring that the sum is scaling. The simplest is to choose $\text{Pr}(p > \alpha) \propto \alpha^{-\nu}$ as the probability of a random duration $p$ exceeding the standard deviation $\Delta R = \langle p \rangle$. This means, for a given duration $p$, the pulse intensity $\Delta R$ is of fixed absolute value and of random sign. Here, the letter $\nu$ is used to denote the random variable whose values are $\alpha$, because the Greek "capital rho" is identical to $p$. Note that this model makes the radical assumption that $\langle \Delta R \rangle = \alpha$; this is necessary in order for the sum to be scaling. In one dimension, Mandelbrot (1984b) refers to this simplest rectangular pulse as a "cancelling echo", since the initial impulse $\Delta R$ is "echoed" at a distance $p$ by an equal and opposite "cancelling" impulse.

This model is scaling, with the exponent $H = 1/\alpha$, because increasing length scales by the factor $\lambda$ only changes the intensity of the echoes by the factor $\lambda^{1/\alpha}$. This may be understood by noting that the number of echoes over an interval of length $l$, whose length exceeds $\rho$, is $|\text{Pr}(p > \alpha) = \lambda^{-\nu}|$. Hence, it is invariant under the scale transformations $l \rightarrow \lambda l$, $p \rightarrow \lambda p$. By construction, the increments of the process are hyperbolic of exponent $\alpha$.

As to the increment over the interval $l$, it is the sum of hyperbolic increments and its tail distribution is hyperbolic. However, it does not follow a stable-Levy distribution. The distribution it follows is a new one introduced in Mandelbrot (1984b), for which the condition $\alpha < 2$ is not necessary. The

discrepancies are very interesting, but they are minor for $\alpha < 1$, and need not concern us here.

The scaling argument (although not the hyperbolicity argument) also works when the pulses are no longer rectangular, but smooth or continuous, as long as their length scales as $\lambda$ and their intensity as $\lambda^{1/\alpha}$. A convenient pulse shape used in later sections is $\Delta R \exp(-u/\rho^2)$, where $u$ is the distance from the center of the echo. This shape is somewhat arbitrary, but it is convenient: by varying $s$, the echo can be made as smooth or as discontinuous as desired. In the limit $s \rightarrow \infty$, the pulses are rectangular.

It is important to note that, although smoothing doesn't affect the scaling, it does considerably alter the probability distributions of the (Euclidean) gradients. Only in the case of sharply rectangular pulses ($s = \infty$) does one obtain hyperbolic gradients $q_1 = \alpha$. In the smooth echo case, the Euclidean gradients are rarely large (rapid fall-off in the probability distributions). Fortunately, as we show below, the Lagrangian gradients remain hyperbolic, $q_1 = \alpha$.

Mandelbrot (1984b) also studies more general RSP processes, and other "cancelling echo" processes, and derives expressions for their characteristic functions, showing in particular that increments of the above process are hyperbolic, with $\nu$ as parameter for all $\alpha < 2$. He also studies a model in which $\Delta R$ and $\rho$ are chosen independently, allowing $H < 1/\alpha$.

3.4. Two-dimensional FSP processes

We encounter a considerably richer set of phenomena than in the linear case described above. In this case, the horizontal section can take any desired shape. In particular, we only treat the isotropic case, and closely the procedures that Mandelbrot (1982) derived from the very different problem of the texture of the distribution of galaxies (Chapter 35 of Mandelbrot (1982)).

In 2-D, the simplest pulse is a strip whose base is a circle. In this case, the total area covered by the pulse areas should be $\text{Pr}(A > \alpha) > 1$. Furthermore, $\Delta R = \lambda^{1/\alpha}$. It is clear that random 1-D sections with no gaps processes with pulse lengths distributed uniformly at random in a square $L \times L$ and the central part $l \times l$ (with $l < L$ as before). The result is shown in Fig. 8.

The field $R(x, y)$ cannot be immediately interpreted in terms of a rain rate because of $R(x, y)$ is almost surely negative at all $x, y$.

In practice, a threshold $R_T$ is set, the ground area is measured as the difference $R - R_T$, and the values are reset to zero like in the fractal of Mandelbrot (1982). The model ran.
discrepancies are very interesting, but they are minor for \( n \sim \frac{1}{3} \), and need not concern us here.

The scaling argument (although not the hyperbolicity argument) also works when the pulses are no longer rectangular, but smooth or continuous, as long as their length scales as \( \lambda \) and their intensity as \( \lambda^{3n} \). A convenient pulse shape used in later sections is \( \Delta R \exp \left( -\left( u/\rho \right)^{2n} \right) \), where \( u \) is the distance from the center of the echo. This shape is somewhat arbitrary, but is convenient: by varying \( s \), the echo can be made as smooth or as discontinuous as desired. In the limit \( s \to \infty \), the pulses are rectangular.

It is important to note that, although smoothing doesn’t affect the scaling, it does considerably alter the probability distributions of the (Eulerian) gradients. Only in the case of sharply rectangular pulses (\( s = \infty \)) does one obtain hyperbolic gradients (\( a = a \)). In the smooth echo case, the Eulerian gradients are rarely large (rapid fall-off in the probability distributions). Fortunately, as we show below, the Lagrangian gradients remain hyperbolic, allowing \( H < 1/a \).

Mandelbrot (1984b) also studies more general FSP processes, and other “cancelling echo” processes, and derives expressions for their characteristic functions, showing in particular that increments of the above process are hyperbolic, with \( a \) as parameter for all \( a > 2 \). He also studies a model in which \( \Delta R \) and \( \rho' \) are chosen independently, allowing \( H < 1/a \).

3.3. Computer simulations of the FSP process in one dimension

The implementation is straightforward. We seek a simulation whose minimum resolution \( \rho \), (the “inner scale”) is taken as 1 pixel, and whose “outer scale” is \( \rho_0 \). We model this process by placing pulse centers uniformly at random over the interval \((0, L)\), \( v \) per unit length, with pulse lengths distributed as \( \Pr(p' > \rho) = \rho^{-1} \) for \( \rho > 1 \). The amplitudes are simply \( \Delta R = \pm \rho^{1/a} \). Such samples exhibit “edge” effects due to the fact that the probability of finding a large jump is smaller near the edges than in the center, because large jumps’ centers are far away. However, the sample’s center of length \( l < L \) will be scaling, with the parameter \( H = 1/a \) up to the outer scale \( \rho_0 = vL \). (Because on average, this will be the longest pulse.) In practice, \( L/l = 6 \) was usually used although \( L/l < 3 \) would probably suffice. Fig. 7 shows an example of the FSP process with 50,000 pulses, \( a = \frac{1}{3} \).

3.4. Two-dimensional FSP processes

We encounter a considerably richer structure in the \( x,y \) plane than in the linear cross sections described above. In this case, the horizontal cross section can take any desired shape. In the following, we only treat the isotropic case, and we follow closely the procedures that Mandelbrot (1984b) derived from the very different problem of the texture of the distribution of galaxies studied in Chapter 35 of Mandelbrot (1982).

In 2-D, the simplest pulse is a straight cylinder whose base is a circle. In this case, the distribution of the pulse areas should be \( \Pr(A > a) = a^{-1} \), for \( a > 1 \). Furthermore, \( \Delta R = \pm \rho^{1/a} \). It is not hard to show that random 1-D sections will be FSP processes with pulse lengths distributed as \( \rho^{-1} \), as before. The centers of the circles are now placed uniformly at random in a square \( L \times L \), and only the central part \( l \times l \) (with \( l < L \) as before) is used. The result is shown in Fig. 8a.

The field \( R(x,y) \) cannot be immediately interpreted in terms of a rain rate because the value of \( R(x,y) \) is almost surely negative at some points. In practice, a threshold \( R_c \) is set, the rain rate is measured as the difference \( R - R_c \), and negative values are reset to zero like in the fractal mountain of Mandelbrot (1982). The model rain rate is then
pulses having a smooth cross section (see Fig. 10 for an intermediate stage of an annulus model).

If we take a unit annulus with an inner radius $\Lambda$, area $\pi$, then

$$\frac{\Lambda^2 - \Lambda^2}{\Lambda} = 1,$$  

hence $\Lambda^* = \Lambda$. 

Furthermore, the mean radius $\bar{\delta}$ and mean radius $\sigma$ satisfy

$$\delta = \frac{1}{\Lambda} \Lambda^* \Lambda,$$  

$$\sigma = \frac{1}{\Lambda - \Lambda^*}.$$  

Increasing $\Lambda$ causes the annulus to become ever smaller ($\Lambda \to 0$). The areas of annuli are chosen so that $\Pr(A < a) \propto a^{-1}$. Fig. 11 shows the distribution of rain率为 $A$ on a black background, representing the appearance of clouds. As $\Lambda$ increases from 1 (circles) to $\Lambda = 1.4$ (thin annuli), the rain structures become increasingly fringed and other new structures appear. The rain-free regions (lacunas) become ever wider. The number of small rain areas increases to 0.

$\Delta r$ is taken as a measure of the size of the rain area.

Fig. 9. The distribution of model rain-rate gradients $\Pr(\Delta r > \Delta r)$. The discontinuous ($s = \infty$) case in Fig. 7a is shown as triangles, the continuous ($s = 8$) case in Fig. 7a is represented by open circles, and the data for 0.25 km (Fig. 4a) is shown by closed circles. The model curves are in arbitrary units and have been separated by half an order of magnitude for clarity of presentation. The straight line indicates a function of the form (\(\Delta r\))

The $s = \infty$ curve deviates from this at large $\Delta r$ because the thresholding of low rain rates systematically truncates the very large $\Delta r$ values.

As a final example of what can be achieved by varying the basic pulse shape, the contour replaced by ellipses with varying eccentricity. Fig. 13 shows a case with high eccentricity. The circular base model, the ellipses can be used to simulate the effect of varying $\Lambda$, $\Lambda = 1$ to $\Lambda = 1.25$.

3.5. Three-dimensional FSP process: simulate temporal evolution

We have shown how to use a FSP model of continuous pulses to simulate the 2-D process, including the probability distribution of $\Delta r$ (with $H_1 = 0.50$ and the mean $\Delta r = 0.50$). We now show that this method can be generalized to simulate the 3-D process. To do this, we need to incorporate the effect of the finite correlation length of rain field. The simplest method is to choose a different pulse basis to replace the circles by annuli of equivalent area with
Fig. 9. The distribution of model rain-rate gradients \( \Pr(\Delta R > \Delta r) \). The discontinuous (\( s = \infty \)) case in Fig. 7a is shown as triangles, the continuous (\( s = 8 \)) case in Fig. 7b is represented by open circles, and the data for 0.25 km (from Fig. 4a) is shown by closed circles. The model curves are in arbitrary units and have been separated by half an order of magnitude for clarity of presentation. The straight line indicates a function of the form \((\Delta r)^{-1/2}\). The \( s = \infty \) curve deviates from this at large \( \Delta r \) because the thresholding of low rain rates systematically truncates the very large \( \Delta r \) values.

The straight line indicates a function of the form \((\Delta r)^{-1/2}\). The \( s = \infty \) curve deviates from this at large \( \Delta r \) because the thresholding of low rain rates systematically truncates the very large \( \Delta r \) values.

The “circular base” model has at least two defects. First, the number of separate rain areas is too small to be realistic: it tends to produce one or two very large areas and few small ones. Using a notion introduced in Chapter 34 of Mandelbrot (1982), this model is called “excessively lacunar”. It is desirable to introduce a new procedure with a new parameter to control the lacunarity by breaking all the areas into smaller ones. A second defect is that, although the model can produce line-like structures (the edges of enormous circles), and thus model rain “fronts”, it does not in general produce the bands that are also characteristic of real rain shapes. The simplest method of eliminating both defects is to choose a different pulse basis: to replace the circles by annuli of equivalent area with pulses having a smooth cross section as in Fig. 8b (see Fig. 10 for an intermediate stage of construction of an annulus model).

If we take a unit annulus with an outer radius of \( \Lambda \), area \( \pi \), then

\[
\Lambda^2 - \Lambda^* = 1, \quad \text{hence} \quad \Lambda^* = \sqrt{\Lambda^2 - 1}.
\]

Furthermore, the mean radius \( \delta \) and the annulus width \( \sigma \) satisfy

\[
\delta = 4(\Lambda^* + \Lambda), \quad \sigma = 4(\Lambda - \Lambda^*).
\]

Increasing \( \Lambda \) causes the annuli to become thinner (\( \delta \to 0 \)). The areas of annuli are chosen as before, with \( \Pr(\Delta R > \Delta r) \propto \Delta r^{-1/2} \). Fig. 11 shows the effect of varying \( \Lambda \) (white on black background to simulate the appearance of clouds). As \( \Lambda \) increases from \( \Lambda = 1 \) (circles) to \( \Lambda = 1.4 \) (thin annuli), the large structures become increasingly fragmented and banded and other new structures appear. Large rain-free regions (lacunas) become rare, and the number of small rain areas increases. The value \( \Lambda = 1.2 \) yields fairly realistic looking fields. Fig. 12 shows an example (white on black background to simulate the appearance of clouds) showing the effect of varying \( r, \alpha \) for \( \Lambda = 1.2 \). Note the appearance of bands and other line-like structures.

To facilitate computation, the pulse shape used here is \( \Delta R \exp\left(-a(\rho^2) - \delta^2/a^2\right) \), which yields a smooth annulus of “size” \( \rho' \).

As a final example of what can be achieved by varying the basic pulse shape, the circles can be replaced by ellipses with varying eccentricities. Fig. 13 shows a case with high eccentricity. As with the circular base model, the ellipses can be made into elliptical annuli. Fig. 14 shows the effect of increasing the analogous elliptical lacunarity from \( \Lambda = 1 \) to \( \Lambda = 1.25 \).

3.5. Three-dimensional FSP processes used to simulate temporal evolution

We have shown how to use a FSP process with continuous pulses to simulate the 2-D rain field, including the gradient probability distributions (with \( H_s = 0.60 \) instead of the measured value of 0.50). We now show that this model also reproduces the basic hyperbolic distribution of fluctuations in total rain flux from isolated storms. In order to proceed, we clearly need to produce a time series of 2-D rain fields. The simplest way to do this is to assume that the time coordinate has the same properties as the \( x,y \) coordinates. This
assumption is the Taylor hypothesis of "frozen" turbulence. The exact limits of its validity in the atmosphere are not clear, but Zawadzki (1973) has verified it directly from at least 5 to 40 min with radar rain data, and Brown and Robinson (1979) have shown that it holds up to at least 1000 km for wind fluctuations. This large range of applicability of this hypothesis is directly related to the fact that for atmospheric dynamics, the Kolmogorov -\frac{2}{3} power law spectrum probably holds out to thousands of kilometers in the horizontal, and up to at least 50 h in time (Larsen et al., 1982). Therefore, over this range at least, the first-order statistics of temporal and spectral correlations are the same. In any case, the comparison of a model based on this hypothesis with actual rain data should be illuminating.

Fig. 15 shows a time series made by generalizing the 2-D annular pulses into 3-D spherical shells in the space of coordinates x, y and t, and then making a series of cross-sectional cuts at different values of t. Here the volume \( V \) must have the distribution \( \operatorname{Pr}(V > v) \propto v^{-1} \) and \( \Delta R = \pm V^{1/3} \).

Note that in this model, as in real rain, the largest rain areas have, (i) the longest characteristic time, and, (ii) the largest mean rain rate.

To verify that fluctuations from the fractal noises have hyperbolic fluctuations, we evaluated these fluctuations in exactly the same way as described for the rainfall data (Subsection 5.1); the result is shown in Fig. 16. This is due to the fact that the smoothing used to make the rain rate fluctuations statistically more realistic in 2-D has the effect of biasing the basic hyperbolic intermittency. The model was also recovered for all the different values of m that we have tested. This helps explain why the model is also compatible with rainfall from many types of rain, with different phenomenological regimes. The echo shape is apparently irrelevant, since the temporal intermittency is considered to be independent of the echo shape. An exhaustive study of rainfall geometry is thus necessary to determine which shape is the most realistic under different meteorological conditions.

### 4. Further aspects of fractal noise

#### 4.1. The fractal dimension of rain perimeters

To see whether or not the fractal dimension \( D \) of the rain perimeters in the model as in real rain, it is necessary to determine the fractal dimension \( D \) of the rain perimeters. The theoretical determination of this \( D \) is related to the notion of the "zero-crossing" problem of probability, which is under certain conditions (e.g. so...
Fig. 11. The effect of changing $\Lambda$ on a $400 \times 400$ grid. Here, the log rain intensity is shown as a "white" scale on a black background. The resemblance to cloud shapes is not accidental, since cloud shapes are similar to rain shapes in several ways (see Lovejoy (1982), Lovejoy et al. (1983) for some empirical comparisons). In these illustrations, $s = 4, L/l = 3, \rho_o = 1200, \nu = 10, \Lambda = 1$; (a), $\Lambda = 1.2$; (b), $\Lambda = 1.7$ (c).

\[ \text{distribution} \ \Pr(V > v) \propto v^{-1} \ \text{and} \ \Delta R = \pm v^{3/\alpha}. \]

Note that in this model, as in real rainfall, the largest rain areas have, (i) the longest characteristic time, and, (ii) the largest mean rain rate.

To verify that fluctuations from the fractal storms have hyperbolic fluctuations, we evaluated these fluctuations in exactly the same way as described for the rainfall data (Subsection 2.1). The result is shown in Fig. 16. This shows that the smoothing used to make the rainfall visually and statistically more realistic in 2-D has not altered the basic hyperbolic intermittency. The value $\alpha = \frac{3}{2}$ is recovered for all the different values of $s, \nu$, and $\Lambda$ that we have tested. This helps explain how $\alpha = \frac{3}{2}$ is compatible with rainfall from many regions of the world with different phenomenologies: the basic echo shape is apparently irrelevant as far as the temporal intermittency is concerned. Note that $H_l$ is also close to the observed value, which is $0.72 \pm 0.5$—see Section 4. However, as shown in Section 2, other aspects of the rainfield geometry are quite dependent on the echo shape. A much more exhaustive study of rainfield geometry is obviously necessary to determine which shapes are the most realistic under different meteorological conditions.

4. Further aspects of fractal meteorology

4.1. The fractal dimension of FSP rainfall perimeters

To see whether or not the shapes are as complex in the model as in real rain, it is of interest to determine the fractal dimension $D$ of the model's rain perimeters. The theoretical determination of this $D$ is related to the notoriously difficult "zero-crossing" problems of probability theory. Under certain conditions (e.g. see Mandelbrot
one finds that \( H + D = 2 \). Although we advocate caution in applying this rule, we have numerically confirmed that it holds for FSP processes. Thus, for \( a = \frac{1}{2} \) and \( H = \frac{1}{2} \), it predicts \( D = \frac{3}{2} = 1.40 \). Hetschel and Procaccia (1984) have used \( H + D = 2 \) to estimate the dimension of cloud perimeters as \( D = 1.38 \).

4.2. The distribution of FSP rain areas

A general property of fractal fields (Mandelbrot, 1977, 1982) is that \( \text{Pr}(A > a) \propto a^{-D} \) for the distribution of the areas \( A \) bounded by fractal curves, with a positive parameter \( B \). Any other form of the probability distribution would involve the introduction of a length scale and would break the scaling. In the quasi-Gaussian case (e.g. for islands in the ocean) one finds theoretically that \( B = 1.4D \). Here, we did not attempt to derive the value of \( B \) theoretically but evaluated it numerically, obtaining \( B = 0.5 \pm 0.5 \). This value is somewhat lower than that found empirically: Fig. 17 shows rain area distributions obtained from near instantaneous radar pictures showing \( B \sim 0.75 \). More realistic models, with lower values of \( H_a \), to be studied elsewhere, may lead to an increase in \( B \).

The area distribution provides a convenient way to study the interaction of space and time scales—in particular, the effect of “pooling” rain areas over many consecutive radar maps—as is usually the practice when calculating empirical rain area distributions (e.g. Lopez, 1977). In the model, the effect of pooling data from consecutive \( x, y \) slices is relatively easy to understand because of the complete equivalence of space and time (1 model unit of time = 1 model unit of space). Pooling areas over model time \( T \) clearly introduces a length scale \( L = T \).

Fig. 18 shows the effect of pooling rain areas for a \( 120 \times 120 \) simulation every \( \Delta T = 4 \) time units for a total of 80 units (i.e. \( N = 20 \) consecutive simulations). The \( a^{-0.3} \) behavior extends to areas of \( \sim 200 \) (pixel\(^2 \)) after which a transition occurs. Simulations show that the transition depends on \( \Delta T, N \) as well as on other conditions, for example, the fractal dimension covered by rain at the beginning of the pooling and even on the initial random “seed” of the long sample time series are necessary for this pooling effect. In real data, the process is undoubtedly complicated because the space-time relationship is undoubtedly less simple than in the model. Further, the space/time conversion factor is not known and may vary. The empirical distributions always involve pooling to increase the number of events, and hence their interpretation, is not straightforward. Also shown in Fig. 16 is the empirical distribution replotted from Houze et al. (1977) obtained from one scan per day over the entire 90 day GATE experiment. Notably, the \( a^{-0.73} \) behavior extends out to \( a \sim 10 \)
FRACTAL PROPERTIES OF RAIN, AND A FRACTAL MODEL

Fig. 13. A series showing the elliptical model on a 400 x 400 grid, $\alpha = \frac{1}{3}$, $s = 2$, $v = 0.25$, $\Lambda = 1$, $\rho_0 = 1200$, and a high average eccentricity. Fig. 14a shows an intermediate case in the same series.

taneous radar pictures showing $B \approx 0.75$. More realistic models, with lower values of $H_v$, to be studied elsewhere, may lead to an increase in $B$.

The area distribution provides a convenient way to study the interaction of space and time scales—in particular, the effect of “pooling” rain areas over many consecutive radar maps—as is usually the practice when calculating empirical rain area distributions (e.g., Lopez, 1977). In the model, the effect of pooling data from consecutive $x$, $y$ slices is relatively easy to understand because of the complete equivalence of space and time (1 model unit of time = 1 model unit of space). Pooling simulated areas over model time $T$ clearly introduces a length scale $L = T$.

Fig. 18 shows the effect of pooling rain areas from a $120 \times 120$ simulation every $\Delta T = 4$ time units for a total of 80 units (i.e. $N = 20$ consecutive simulations). The $a^{-0.3}$ behavior extends to areas of $\sim 200$ (pixel)$^2$ after which a rapid fall-off occurs. Simulations show that the exact point of transition depends on $\Delta T$, $N$ as well as the initial conditions, for example, the fraction of area covered by rain at the beginning of the sequence, and even on the initial random “seed.” Extremely long sample time series are necessary to remove this pooling effect. In real data, the situation is complicated because the space-time relationship is undoubtedly less simple than in the model, and the space/time conversion factor is not known and may vary. The empirical distributions almost always involve pooling to increase the data base; hence their interpretation, is not straightforward. Also shown in Fig. 16 is the empirical GATE distribution replotted from Houze and Cheng (1977) obtained from one scan per day over the entire 90 day GATE experiment. Note that the $a^{-0.75}$ behavior extends out to $a \sim 10^4$ km$^2$. The
truncation for a larger probability indicates that the sampling was over an insufficiently long time period, although the limited area of reliable radar coverage (~50,000 km²) could also account for the observed fall-off.

4.3. The elliptical dimension of atmospheric motions

At small scales, atmospheric motions appear three-dimensional and at large scales they appear two-dimensional. It has often been conjectured that a “dimensional transition” separates the small and large scale motions. However, Schertzer and Lovejoy (1984a, b) propose a model in which the atmosphere is neither two- nor three-dimensional, in scaling, but anisotropic.

In this model, the horizontal motions are governed primarily by energy transfer, yielding $H = \frac{1}{3}$ in the horizontal, and the vertical motions primarily by the buoyancy force variance transfer, yielding $H = \frac{1}{3}$ in the vertical. These very different scaling exponents lead to the phenomenon of “stochastic stratification”, i.e., to fields that appear increasingly stratified at larger and larger scales according to a well-defined scaling mechanism characterized by an “elliptical dimension” (Schertzer and Lovejoy (1985)) $D_E = 2 + \frac{(4)}{(4)} = \frac{4}{3} = 2.555 \ldots$ Using this effective dimension to express anisotropy yields plausible vertical cross sections of rain fields (see, Schertzer and Lovejoy (1984a)).

4.4. The transition from meteorology to climatology

Lovejoy and Schertzer (1983, 1984) examine temperature and paleotemperature series and find (ignoring the diurnal cycle and its harmonics) that local temperatures are scaling up to periods of ~1 month (the “synoptic maximum”), after which they become stationary. The average hemispheric temperatures also appear to be scaling from ~40,000 years down to at least 5 years with almost exactly the same value of $H$ (~0.4). To model this behavior, the authors assume that the fundamental differences between meteorological and climatological temperatures are the length and time scales over which they are averaged. They speculate that the break in the scaling for local temperatures at ~1 month would disappear in true globally averaged temperatures. Although these results are preliminary, it would be interesting to determine whether averaging over larger and larger areas affects the spectrum of rainfall fluctuations in the same way.

5. Discussion. Comparison of the fractal models, including FSP, with earlier models of rainfall

The literature contains a variety of stochastic models of rainfall based upon the assumption that not only the physics but also the rainfall take different forms over different scales and time ranges. For reviews of this general approach especially Waymire and Gupta (1984), Litube et al. (1984) or Rodriguez-Iturbe (1984).

This approach subdivides different scales and the geometry of rainfall is subdivided into “small scale”, “large meso-scale”, and “small area” and “cells.” It is assumed that rainfall in these ranges, the autocorrelation function, like a Markov process, does not follow standard methods of analysis but is fractal. Unfortunately, in order to achieve this, it is observed in rain “events” that is necessary to subordinate the small scale to the large scale processes. This raises profound difficulties, how wide-recognized by practitioners, but implementations are artificial. In practical application involves large numbers.
Fig. 14b.

Figures also appear to be scaling from ~40,000 years down to at least 5 years with almost exactly the same value of $H$ (~0.4). To model this behavior, the authors assume that the fundamental differences between meteorological and climatological temperatures are the length and time scales over which they are averaged. They speculate that the break in the scaling for local temperatures at ~1 month would disappear in true globally-averaged temperatures. Although these results are preliminary, it would be interesting to determine whether averaging over larger and larger areas affects the spectrum of rainfall fluctuations in the same way.

5. Discussion. Comparison of the fractal models, including FSP, with earlier models of rainfall

The literature contains a variety of stochastic models of rainfall based upon the assumption that not only the physics but also the geometry of rainfall take different forms over different scale ranges. For reviews of this general approach, see especially Waymire and Gupta (1981), Rodriguez-Iturbe et al. (1984) or Rodriguez-Iturbe (1983).

This approach subdivides difficult problems into portions expected to be more manageable. Thus, the geometry of rainfall is subdivided into “synoptic”, “large meso-scale”, and “small meso-scale” areas and “cells”. It is assumed that, within each of these ranges, the autocorrelation function is exponential, like a Markov process so that many standard methods of analysis become available. Unfortunately, in order to achieve the clustering that is observed in rain “events”, it becomes necessary to subordinate the small-scale processes to the large-scale processes. This subdivision raises profound difficulties, however, and it is widely recognized by practitioners that actual implementations are artificial. In particular, implementation involves large numbers of parameters.

Lovejoy (1985), 3
They are to be determined by using rainfall data at specific time and space scales, and none of them appears to reflect in any direct fashion the basic physics of the rainfall process. Moreover, when the resulting process is used to generate synthetic storms (e.g., in hydrological modeling), the results of extrapolation are poor except for the range of scales for which the model had been calibrated.

Thus, the most persuasive argument for distinguishing several ranges of scaling in the atmosphere resides in the smallness of the scale height in the atmospheric mean pressure field (~10 km) compared with its width. Nevertheless, these vertical fluctuations do not appear to have a scale height, and (Subsection 4.3) the differences between the horizontal and vertical scaling laws can be accounted for.

To attack numerous analogous problems that occur in other areas of science, the theory of fractals forsakes subdivision. This approach had at one time seemed doomed to failure, yet it has repeatedly turned out to be successful on the following major grounds. (1) Fractals provide a simple and elegant phenomenological description of a broad class of anthropogenic problems using a very small number of parameters. (2) These parameters can be estimated accurately and with high precision, and can be related to other—more familiar—quantities of interest. (3) Furthermore, the available fractal description has invariably suggested the development of new theoretical investigations that have had an impact on the fractal assumptions and led to substantive progress.

Most relevant from the viewpoint of this work is turbulence, which is ruled by the same equations as those of Navier-Stokes. Since the Navier-Stokes equations involve no scale above a very small cutoff associated with viscosity, Manneville and Ruelle have recourse to the so-called "principle of microscopic self-similarity" when a problem is "symmetric" (that is, when it is invariant under a class of transformations which preserves..."
ted in a different fashion, by an anisotropic fractal model of dimension 2.555.

To attack numerous analogous problems that occur in other areas of science, the theory of fractals forsakes subdivision. This approach had at one time seemed doomed to failure, yet it has repeatedly turned out to be successful on the following major grounds. (1) Fractals provide a successful phenomenological description of a broad class of phenomena using a very small number of parameters. (2) These parameters can be estimated consistently and with high precision, and can be related to other—more familiar—quantities of interest. (3) Furthermore, the availability of a fractal description has invariably soon triggered theoretical investigations that have justified the fractal assumptions and led to substantial overall progress.

Most relevant from the viewpoint of meteorology is turbulence, which is ruled by the same equations, those of Navier-Stokes. Since the Navier-Stokes equations involve no scale above a very small inner cutoff associated with viscosity, Mandelbrot had recourse to the so-called "principle of symmetry": when a problem is "symmetric" (that is, is invariant under a class of transformations which need not be
left and right symmetry), the problem's solution should also be expected to be symmetric. Exceptions do occur, when a symmetric solution would be against other laws of nature, in which case a "symmetry breakdown" occurs. As applied to the Navier–Stokes equation, (most explicitly in Chapter 11 of Mandelbrot (1982)), this argument led Mandelbrot to the hypothesis that the phenomenon of turbulence is a reflection of scaling singularities of the Navier–Stokes equations and of their Euler limit for zero viscosity. This hypothesis has proven very fruitful, and many writers believe it is in the process of being confirmed. While skeptics may still consider that such a bold conjecture needs further confirmation, there is no question that the fractal approach has brought a fresh perspective and new activity in the study of Navier–Stokes equations.

Our hope is that the same will be the case to some extent with the models presented in this paper. Further encouraging features reside in the fact that the boundary conditions of weather include many fractals: such as the shape of the mountains and coastlines (Mandelbrot, 1975, 1977, 1982), and atmospheric forcing (solar insolation), (private communication from C. Gauthier, 1984).

6. Conclusions

We have proposed a fractal model for the static structure and the temporal evolution of the horizontal rainfall field based on three basic parameters: the scaling parameter $H_0$, the intermittency parameter $A$, and the lacunarity parameter $\lambda$. Two other parameters—the smoothness parameter $s$, and the pulse density parameter $v$, were also introduced but could not be determined very accurately; within a wide range, their influence seems limited to higher-order statistical properties of the model.

The scaling parameter $H_0$ relates the large- and small-scale structure of a fractal field, determined from tropical radar rainfall, a value $H_0 \approx 0.50$. The intermittency is the hyperbolic exponent of the difference in rainfall fluctuations from isolated storm tracks, in Spain, and in the tropical Atlantic, found to have a value $\approx 3$.

In 2-D, the fractal sum of pulsation fields of Mandelbrot (1984c) consists of a circular annular shaped unit fluctuations. It produces a scaling fractal rain model with $H_0 = 1/\alpha = 0.60$. By making the advection field progressively thinner (increasing $\lambda$) the model
6. Conclusions

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The scaling parameter $H$ relates the large- and small-scale structure of a fractal field, and was determined from tropical radar rainfall data to have a value $H \approx 0.50$. The intermittency parameter $\alpha$ is the hyperbolic exponent of the distribution of rainfall fluctuations from isolated storms. In Montreal, in Spain, and in the tropical Atlantic, it was found to have a value $\approx 0.5$.

In 2-D, the fractal sum of pulses model of Mandelbrot (1984c) consists of a hierarchy of annular shaped unit fluctuations. It was used to produce a scaling fractal rain model with $\alpha = \frac{1}{2}$ and $H = 1/\alpha = 0.60$. By making the annuli progressively thinner (increasing $A$) the model's struc-
FRACTAL PROPERTIES OF RAIN, AND A FRAC TAL MODEL

Table 1. Various fractal parameters estimated from the data, and the corresponding simulation based upon the estimated α and A

<table>
<thead>
<tr>
<th>Data</th>
<th>α</th>
<th>H_k</th>
<th>H_k</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.66 ± 0.05</td>
<td>0.50 ± 0.02</td>
<td>0.64 ± 0.04</td>
<td>0.75 ± 0.05</td>
</tr>
<tr>
<td>Model</td>
<td>0.5</td>
<td>0.5</td>
<td>0.72 ± 0.03</td>
<td>0.50 ± 0.03</td>
</tr>
</tbody>
</table>

1 GATE data yield 0.48 ± 0.02, and Montreal data yield 0.50 ± 0.05.
2 Putting together data for GATE, Moscow, and for Spain.
3 GATE data only.
4 GATE and Indian Ocean data yield D = 1.35 (Lovejoy, 1982), data over France yield D = 1.30 (1983), and data over the Mediterranean yield D = 1.345 (Cahalan, private communication).
5 The row marked “model” was obtained with α = 0.5, H_k = 0.5, A = 1.2, ρ = 1, s = 2, p = /t indicated under 5 or 6. Over a considerable range, the parameters were insensitive to the value of B (e.g. 1.0 < B > 0.1, 12 > s > 1), and, with the exception of D, insensitive to changes in A. The affect the lacunarity, but a quantitative investigation of this parameter is outside the scope of the paper; these parameters are expected to be independent of R_t, i.e., the coverage, although this, for example, statistics, may not be true.
6 This value was estimated from the fluctuations in model rain from the evolution of 1-D convective rolls with s = ∞ (the 1-D analog of the 2-D “Lagrangian”) H_k described in Subsection 2.2). This 1-D model is far easier to calculate than the true 2-D value. s = ∞ was chosen rather than s = 2 because of its hyperbolic (α = 0.5), while the latter was not.
7 D = 2 − 1/α, as described in Subsection 4.1.

and the existence of “fronts” and “bands” at all scales. The model is very flexible in that it apparently yields scaling and the correct hyperbolic temporal intermittency irrespective of the basic shape. The constraint H = 1/α can be lifted to allow H > 1/α, but this requires a generalization of the FSP process, which cannot be dealt with here.

Other possibilities for extending and improving the model include modeling the vertical structure by using the notion of “stochastic stratification” and taking the Coriolis force into account (“stochastic zonality”), as in Schertzer and Lovejoy (1984a). It will also be interesting to relate the model parameters to those of other meteorological fields.

Finally, this model may prove useful in the modeling of other hyperbolically intermittent scaling fields, as Schertzer and Lovejoy argue for the case of the wind, temperature, and energy flux fields.

7. Acknowledgements

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Table 1. Various fractal parameters estimated from the data, and the corresponding values in the simulation based upon the estimated $\alpha$ and $\Lambda$

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$H_E$</th>
<th>$H_I$</th>
<th>$B$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>$1.66 \pm 0.05^2$</td>
<td>$0.50 \pm 0.02^3$</td>
<td>$0.64 \pm 0.05^4$</td>
<td>$0.75 \pm 0.05^5$</td>
<td>$1.35^6$</td>
</tr>
<tr>
<td>Model$^7$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{4}$</td>
<td>$0.72 \pm 0.05^6$</td>
<td>$0.50 \pm 0.05$</td>
<td>$7/5^7$</td>
</tr>
</tbody>
</table>

1 GATE data yield $0.48 \pm 0.02$, and Montreal data yield $0.50 \pm 0.05$.
2 Putting together data for GATE, Montreal and for Spain.
3 GATE data only.
4 GATE and Indian Ocean data yield $D = 1.35$ (Lovejoy, 1982), data over France yield $D = 1.38$ (Lovejoy et al. 1983), and data over the Mediterranean yield $D = 1.345$ (Cahalan, private communication).
5 The row marked "model" was obtained with $\alpha = 1$, $H_E = \frac{3}{4}$, $\Lambda = 1.2$, $r = 1$, $\sigma = 2$, $\rho_0/l = 3$, except where indicated under 4 or 6. Over a considerable range, the parameters were insensitive to the values of $r$ and of $\sigma$ (e.g. $1 > r > 0.1, 12 > \sigma > 1$), and, with the exception of $D$, were insensitive to changes in $\Lambda$. The value of $\Lambda$ does affect the lacunarity, but a quantitative investigation of this parameter is outside the scope of this paper. All of these parameters are expected to be independent of $R$, i.e., the % coverage, although this, for certain high-order statistics, may not be true.
6 This value was estimated from the fluctuations in model rain from the evolution of 1-D connected sections with $s = \infty$ (the 1-D analog of the 2-D "Lagrangian" $H_E$ described in Subsection 2.2). This 1-D estimate of $H_E$ is far easier to calculate than the true 2-D value. $s = \infty$ was chosen rather than $s = 2$ because the former was hyperbolic ($\alpha = \frac{1}{2}$), while the latter was not.
7 $D = 2 - 1/\alpha$, as described in Subsection 4.1.

and the existence of "fronts" and "bands" at all scales. The model is very flexible in that it apparently yields scaling and the correct hyperbolic temporal intermittency irrespective of the basic shape. The constraint $H = 1/\alpha$ can be lifted to allow $H < 1/\alpha$, but this requires a generalization of the FSP process, which cannot be dealt with here.

Other possibilities for extending and improving the model include modeling the vertical structure by using the notion of "stochastic stratification" and taking the Coriolis force into account ("stochastic zonality"), as in Schertzer and Lovejoy (1984a). It will also be interesting to relate the model parameters to those of other meteorological fields. Finally, this model may prove useful in the modeling of other hyperbolically intermittent scaling fields, as Schertzer and Lovejoy (1984a, b) argue for the ease of the wind, temperature, buoyancy force and energy flux fields.

7. Acknowledgements

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