The emergence of the Climate

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Abstract

The classical laws of turbulence associated with the names Kolmogorov, Corrsin, Obhukov, and Bolgiano were expected to emerge when continuum mechanics was taken to the strongly nonlinear limit. In order to apply to the atmosphere over wide ranges they must be generalized to account for anisotropy and intermittency; over the last twenty five years this has been done. We review the theory and compare it with recent global scale analyses showing that the new laws hold up to planetary scales and account for weather dynamics up to time scales $\tau_w \approx 10$ days (the lifetime of planetary scale structures). We focus on the transition from the weather to the climate at scales $\tau > \tau_w$. By making a third generalization of the classical laws, we show that the spatial interactions become very weak causing a drastic “dimensional transition”: all spectra become a relatively flat “spectral plateau”. The main complication is that high frequency “ocean weather” undergoes the same transition but at scales $\tau_o \approx 1$ year. We show that the “weather-climate” regime beyond $\tau_w$ continues up to scales of $10 - 100$ years giving way to the climate regime proper which continues up to around $100 \text{ kyr}$. Whereas the statistics of the weather-climate regime is accurately predicted as an extension of the basic weather regime, this new climate regime involves new (but still scaling) physics and overall, there are three scaling regimes. This allows us to objectively distinguish the weather from the climate on the basis of its type of scaling variability.
1. Introduction

1.1. What is the climate?

Notwithstanding the explosive growth of climate science over the last twenty years, there is still no clear universally accepted definition of what the climate is – or what is almost the same thing – what the difference is between the weather and the climate. The core idea shared by most climate definitions is famously encapsulated in the dictum: “The climate is what you expect, the weather is what you get” (Farmer’s Almanac). In more scientific language “Climate in a narrow sense is usually defined as the "average weather," or more rigorously, as the statistical description in terms of the mean and variability of relevant quantities over a period of time ranging from months to thousands or millions of years (Intergovernmental Panel on Climate Change. Appendix I: Glossary. Retrieved on 2007-06-01).

One problem with this definition is that it fundamentally depends on subjectively defined space-time averaging scales. While the World Meteorological Organization defines climate as 30 year or longer variability, a period of two weeks to a month is often used to distinguish weather from climate so that even with these essentially arbitrary periods, there is still a range of about 1000 in scale (30 years /2 weeks) which is up in the air. This fuzzy distinction is also reflected in numerical climate modelling since Global Climate Models are fundamentally the same as weather models but at lower resolutions, with a different assortment of subgrid “parametrisations” they are coupled to ocean models. Consequently, whether we define the climate as the long term statistics of the weather, or in terms of the long term interactions of components of the “climate system”, we still need an objective way to distinguish it from the weather.

However, there is yet another problem with the Almanac and allied definitions: they imply that climate dynamics are nothing new that they are simply weather dynamics at long time scales. This seems naïve since we know from physics that when processes repeat over wide enough ranges of space and or time scale we are used to observing that qualitatively new features emerge so that over long enough time scales we expect that qualitatively new climate laws should emerge from the higher frequency weather laws. These qualitatively new “emergent” laws could simply be the consequences long range statistical dependencies of weather physics in conjunction with qualitatively new climate processes; their nonlinear synergy giving rise to emergent laws of climate dynamics.

Since the atmosphere is a nonlinear dynamical system with interactions and variability occurring over huge ranges of space and time scales (millimetres to planet scales, milliseconds to billions of years, ratios $\approx 10^{10}$ and $\approx 10^{20}$ respectively), the natural approach is to consider it as a hierarchy of processes each with wide range scaling, i.e. each with nonlinear mechanisms that repeat scale after scale over potentially wide ranges. Following (Lovejoy and Schertzer, 1986), (Pelletier, 1998; Schmitt et al., 1995), (Koscielny-Bunde et al., 1998), (Talkner and Weber, 2000), (Blender and Fraedrich, 2003), (Ashkenazy et al., 2003), (Huybers and Curry, 2006b) this approach is increasingly superceding earlier approaches that postulated more or less white noise backgrounds with a large number of spectral “spikes” corresponding to many different quasi-periodic processes. This includes the slightly more sophisticated variant (Mitchell, 1976)
which retains the spikes but replaced the white noise with a hierarchy of Ornstein-Uhlenbeck processes (white noises and their integrals); in the spectrum, “spikes” and “shelves”; see also (Fraedrich et al., 2009) for a hybrid composite including a single short range scaling regime.

Over the past 25 years, scaling approaches have been increasingly applied to the atmosphere, especially at meteorological scales, and in the last five years increasingly to global scales. This has given rise to a new scaling synthesis covering the entire gamut of meteorological scales from milliseconds to beyond the $\approx$10 day period which is the typical lifetime of planetary structures i.e. the weather regime, reviewed in (Lovejoy and Schertzer, 2010b). It was concluded that the theory and data were consistent with wide range but anisotropic spatial scaling, and that the lifetime of planetary sized structures, provides the natural scale at which to distinguish weather and climate. In this paper, we retain the same scaling framework, but complement the earlier review by focusing on the longer time scales associated with the weather-climate transition and the climate proper.

The model discussed in this overview is the consequence of a series of generalizations of the classical laws of turbulence associated with Kolmogorov, Corrsin, Obhukhov and Bolgiano. After a quick empirical overview of the scaling using spectral analysis in space (section 2) and in time (section 3), the classical laws are generalized in three important ways. The first is to anisotropy (section 4) necessary at the very least to take into account the scaling vertical stratification which is in turn a sine qua non for wide range horizontal and temporal scaling. The second is the replacement of the classical picture of smooth, quasi Gaussian variability by strong intermittency generated by multifractal cascades (section 5). These generalizations are enough to model the atmosphere over “weather” scales (planetary scales in space and up to $\approx$ 10 days in time). To go beyond this, we need a further generalization to cascade processes whose nondimensional outer time scale is much longer than the corresponding (nondimensional) outer space scale (section 6). This yields a drastic “dimensional transition”; followed by a qualitatively different weather-climate regime which apparently holds up 30 -100 years.

2. A brief review of evidence for wide range spatial scaling using spectra

2.1 Discussion

Although our focus is on time scales longer than the weather scales we first give a quick empirical tour of the spatial, and then temporal scaling of the fields relevant either directly or indirectly to atmospheric dynamics. This overview is by no means exhaustive; it partly reflects the availability of relevant analyses and partly the significance of the fields in question. Spatial scaling is fundamental for weather processes and it is the collapse (“dimensional transition”) of the spatial interactions which distinguishes the high frequency weather from the low frequency weather – climate regime out to 30 – 100 year scales. In order to simplify things as much as possible, in this section we will only use spectra.

Consider a random field $f(\mathbf{r})$ where $\mathbf{r}$ is a position vector (extensions to space-time are straightforward). Its “spectral density” $E(k)$ is the total contribution to the variance of the process due to structures of with wavenumber between $k$ and $k + dk$ (i.e. due to structures of size $2\pi/l$ where $l$ is the corresponding spatial scale. The spectral density thus satisfies:
\[ \left\langle f(\ell)^2 \right\rangle = \int_0^\infty E(k) dk \]  

(1)

where \( \langle f(\ell)^2 \rangle \) is the total variance (assumed to be independent of position; the angular brackets \( < . , > \) indicate statistical averaging).

In the following examples, we demonstrate the ubiquity of power law spectra:

\[ E(k) = k^{-\beta} \]

(2)

If we now consider the real space (isotropic) reduction in scale by factor \( \lambda \) we obtain: \( \ell \rightarrow \lambda^{-1} \ell \) corresponding to a “blow up” in wavenumbers: \( k \rightarrow \lambda k \); power law \( E(k) \) (eq. 2) maintain their form under this transformation: \( \hat{E} \rightarrow \lambda^{-\beta} \hat{E} \) so that \( E \) is “scaling” and the (absolute) “spectral slope” \( \beta \) is “scale invariant”. If empirically we find \( E \) of the form eq. 2, we take this as evidence for the scaling of the field \( f \). For the moment, we consider only scaling and scale invariance under such conventional isotropic scale changes; in section 4 we extend this to anisotropic scale changes.

### 2.2 The horizontal scaling of atmospheric Fields

The main data sources for horizontal scaling are remote sensing, aircraft, in situ networks and reanalyses, each with their own particular advantages and disadvantages. We start by considering global scale satellite radiances since they are the most straightforward to interpret. Fig. 1 shows the “along track” 1-D spectra from the VIRS (Visible Infra Red Sounder) instrument of the TRMM (Tropical Rainfall Measurement Mission) at wavelengths of 0.630, 1.60, 3.75, 10.8, 12.0 \( \mu m \) i.e. for visible, near infra red, and (the last two) thermal infra red. Each channel was recorded at a nominal resolution of 2.2 \( km \) and was scanned over a “swath” 780 \( km \) wide and \( \approx 1000 \) orbits were used in the analysis. The scaling apparently continues from the largest scales (20,000 \( km \)), to the smallest available. The scaling observed in the visible channel (1) and the thermal IR channels (4, 5) are particularly significant since they are representative respectively of the energy containing short and long wave radiation fields which dominate the earth’s energy budget. Thanks to the effects of cloud modulation, the radiances are very accurately scaling. This result is incompatible with classical turbulence cascade models which assume well-defined energy flux sources and sinks with a source and sink - free “inertial” range in between). Also of interest is the fact that the spectral slope \( \beta \) value is close (but a little lower) than the value \( \beta = 5/3 \) expected for passive scalars in the classical Corrsin-Obukhov theory. This result is consistent with theoretical studies of radiative transfer through passive scalar multifractal clouds ((Watson et al., 2009), (Lovejoy et al., 2009c)). Although we cannot directly interpret the radiance spectra in terms of the wind, humidity or other atmospheric fields, they are strongly nonlinearly coupled to these fields so that the scaling of the radiances are prima facie evidence for the scaling of the other fields. To put it the other way around: if the dynamics was such that it primarily produced structures at a characteristic scale \( L \), then it is hard to see how this scale would not be clearly visible in the associated cloud radiances.
Fig. 1: Spectra from ≈ 1000 orbits of the VIRS instrument (Visible Infrared Scanner) on the TRMM satellite channels 1-5 (at wavelengths of 0.630, 1.60, 3.75, 10.8, 12.0 µm from top to bottom, displaced in the vertical for clarity). The data are for the period January through March 1998 and have nominal resolutions of 2.2 km. The straight regression lines have spectral exponents $\beta = 1.35, 1.29, 1.41, 1.47, 1.49$ respectively, close to the value $\beta = 1.53$ corresponding to the spectrum of passive scalars (= 5/3 minus intermittency corrections, see ch.3). The units are such that $k = 1$ is the wavenumber corresponding to the size of the planet (20000 km$^{-1}$). Channels 1, 2 are reflected solar radiation so that only the 15600 km sections of orbits with maximum solar radiation were used. The high-wavenumber falloff is due to the finite resolution of the instruments. Adapted from (Lovejoy et al., 2008b).

If we are interested in the usual meteorological variables of state rather than radiances we may use reanalyses. Reanalyses are hybrid products in the sense that they combine data with constraints imposed by the numerically discretized atmospheric equations in order to determine an optimum estimate of the state of the atmosphere. Fig. 2 shows representative reanalyses taken from the European Medium Range Weather Forecasting Centre (ECMWF) “interim” reanalysis products, the zonal and meridional wind, the geopotential height, the specific humidity, the temperature, vertical wind. They are publicly available at 1.5° resolution in the horizontal and at 37 constant pressure surfaces (every 25 mb in the lower atmosphere), every 6 hours from 1989 - present. The data in fig. 2 were taken from the data rich 700 mb level. It also suffers little from the extrapolations necessary to obtain global 1000 mb fields (which is especially problematic in mountainous regions); see (Berrisford et al., 2009).

We used daily data for the year 2006 concentrating on the band between ±45° latitude (with a cylindrical projection) allowing us to conveniently compare the statistics in the east-west and north-south directions in order to study the statistical anisotropies between the two. The east-west direction was broken up into 2 sections, one from 0-180° and the other from 180-0° longitude. For technical reasons discussed in (Lovejoy and Schertzer, 2011), the spectrum was
estimated by performing integrals around ellipses with aspect ratios 2:1 (EW:NS) the wavenumber scale in fig. 2 indicates the east-west scale.

From the figure we can see excellent scaling except for generally small deviations at the largest scales ($\geq 5000 \ km$), although for the geopotential, the deviations begin nearer to 2500 $km$. While the scaling is quite convincing, the values of the exponents are not “classical” in the sense that they do not correspond to the values predicted by any accepted turbulence theory. An exception is the value $\beta \approx 1.6$ for the humidity which is only a bit bigger than the Corrsin-Obukhov passive scalar value $5/3$ (minus intermittency corrections which for this are of the order of 0.15; see table 1, the intermittency parameters $C_1$, $\alpha$ are described in section 5, the nonconservation parameter $H$ in section 4), although in any case classical (isotropic) turbulence theory would certainly not be expected to apply at these scales. We could also mention that classically, the atmosphere is “thin” at these scales (since the horizontal resolution $\approx 166 \ km$ is much greater than the exponential “scale height” $\approx 10 \ km$), hence according to the classical isotropic 3D/2D theory one would expect 2D isotropic turbulence to apply. This leads to the predictions $\beta = 3$ for the horizontal wind field (a downscale enstrophy cascade) and $\beta = 5/3$ (with an upscale energy cascade). In comparison, we see that the actual value for the zonal wind ($\beta = 2.35$) is in between the two. In (Lovejoy et al., 2009b) we argue that this is an artefact of using gradually sloping isobars (rather than isoheights) in a strongly anisotropic (stratified) turbulence. These spectra already caution us that reanalyses must be validated through scale by scale statistical intercomparisons with other data.

Fig. 2: Inter-comparisons of the spectra of different atmospheric fields from the ECMWF interim reanalysis. Top (red) is the geopotential ($\beta = 3.35$), second from the top (green) is the zonal wind
(β = 2.40), 3rd from the top (cyan) is the meridional wind (β = 2.40), 4th from the top (blue) is the temperature (β = 2.40) 5th from the top (orange) is the vertical wind (β = 0.4), at the bottom, (purple) is the specific humidity (β = 1.6). All are at 700 mb and between ±45° latitude, every day in 2006 at GMT. The scale at the far left corresponds to 20000 km in the east-west direction, at the far right to 660 km. Note that for these 2-D spectra, Gaussian white noise would yield β = -1 (a positive slope = +1).

Finally, let us consider in situ aircraft data, fig. 3; we see that the scaling for the temperature, humidity and potential temperature is excellent over the larger range 560 m to 1140 km. The data are from a scientific aircraft (Gulstream 4) and were taken at 1 Hz (≈ 280 m resolution), the data analyzed in fig. 3 are averages over 24 roughly straight legs, each with 4000 points long (≈ 1120 km), see (Lovejoy et al., 2009b), (Lovejoy et al., 2010).

Fig. 3: Aircraft spectra of temperature (blue, bottom), humidity (red, middle), log potential temperature (gold, top), reference lines β = 2. These are averages over 24 isobaric aircraft “legs” near 200 mb taken over the Pacific Ocean during the Pacific Winter Storms 2004 experiment, the resolution was 280 m (Nyquist wavenumber = (560 m)⁻¹. Adapted from (Lovejoy et al., 2010).

2.3 The atmosphere in the vertical

In spite of the fact that gravity acts strongly at all scales, the classical theories of atmospheric turbulence have all been quasi-isotropic in either two or three dimensions. While a few models tentatively predict possible transitions in the horizontal (for example between $k^{5/3}$ and $k^3$ spectra for a 3D to 2D transition for the wind), in contrast, in the vertical any 3D/2D “dimensional transition” would be even more drastic (Schertzer and Lovejoy, 1985b). This is also true for passive scalars. It is therefore significant that wind spectra from relatively low resolution (50 m) radiosondes are scaling over nearly the entire troposphere (up to 13.3 km) with slope near (but a little larger) than that predicted by Bolgiano and Obukhov (11/5) (Lazarev et al., 1994). Using structure function analyses (section 4), this result has been extended down
from $\approx 100 \text{ m}$ down to $5 \text{ m}$ resolutions with data from state-of-the-art drop sondes in (Lovejoy et al., 2007). The empirical vertical spectral slope value $\beta_v \approx 2.4$ is greater than the horizontal spectral slope value $\beta_h \approx 5/3$; although it is not obvious, this implies in fact that the atmosphere is more and more stratified at larger and larger scales. The critical exponent ratio $(\beta_h - 1)/(\beta_v - 1) = H_z$ is quite near the theoretical value $5/9$ discussed in section 4, the “23/9D” model, so-called because the volume of typical stratified structures increases as the horizontal scale to the power $23/9 \approx 2.55$. It is the fact that the vertical and horizontal scalings are different that allows the horizontal scaling to operate over huge ranges.

2.4 The spatial spectral scaling of the atmospheric boundary conditions

The basic equations of the atmosphere are scaling, indeed, they have recently been shown theoretically to be scaling under anisotropic scale changes (Schertzer et al., 2010), so that solutions can potentially also be scaling with different horizontal and vertical exponents. However, for this to be true, the boundary conditions must be scaling. Perhaps the most significant boundary condition is the topography, including its prime importance for surface hydrology and oceanography, and therefore the interactions hydrosphere-atmosphere. Scaling in topography has a long history, going back nearly 100 years to when (Perrin, 1913) argued that the coast of Brittany was nondifferentiable. Later, (Steinhaus, 1954) speculated on the nonintegrability of the river Vistula, and (Richardson, 1961) quantified both aspects using scaling exponents and (Mandelbrot, 1967) interpreted the exponents in terms of fractal dimensions. Indeed, scaling in the earth’s surface is so prevalent that there are entire scientific specializations such as river hydrology and geomorphology which abound in scaling laws of all types (for a review see (Rodriguez-Iturbe and Rinaldo, 1997).

The first spectrum of the topography was the very low resolution one computed in (Venig-Meinesz, 1951) who already noted that it was nearly a power law with $\beta \approx 2$. After this pioneering work, Balmino et al. (1973) made similar analyzes on more modern data sets and confirmed Vening Meinesz’s results. Bell (1975) followed, combining various data sets (including those of abyssal hills) to produce a composite power spectrum that was scaling over approximately 4 orders of magnitude in scale (also with $\beta \approx 2$). More recent spectral studies of bathymetry over scale ranges from 0.1 km to 1000 km can be found in Berkson and Matthews (1983) ($\beta \approx 1.6$–1.8), Fox and Hayes (1985) ($\beta \approx 2.5$), Gibert and Courtillot (1987) ($\beta \approx 2.1$–2.3) and Balmino (1993) ($\beta \approx 2$). Using the modernETOPO5 data set (earth’s topography including bathymetry) at 5 minutes of arc (roughly 10 km) along with other higher resolution but regional Digital Elevation Models (DEM’s) (Gagnon et al., 2006) found that the spectrum follows a scaling form with $\beta \approx 2.1$ down to at least $\approx 40 \text{ m}$ in scale (see this reference also for multifractal analyses.

Another important atmospheric boundary is the ocean surface; in section 4.3 we discuss the scaling of currents and sea surface temperatures (SST) which are also argued to be scaling up to planetary scales. Other surface fields are also scaling over wide ranges for example, remotely sensed vegetation and soil moisture indices (Lovejoy et al., 2007).
3. Temporal scaling, weather and climate

3.1 Temporal spectral scaling in the weather regime

Temporal scaling and its limits is the main subject of this overview, let us consider evidence for its ubiquity. One of the earliest was that of (Van der Hoven, 1957) which is at the origin of the famous “meso-scale gap”, the supposedly energy-poor spectral region between roughly 10-20 minutes and \( \approx 4 \) days (ignoring the diurnal spike). Even until fairly recently, textbooks regularly reproduced the spectrum (often redrawing it on different axes or introducing other adaptations), citing it as convincing empirical justification for the neat separation between low frequency isotropic 2D turbulence - identified with the weather - and high frequency isotropic 3-D “turbulence”. If the gap were real, the turbulence would be no more than an annoying source of perturbation to the (2-D) weather processes.

However, it was quickly and strongly criticized (e.g. (Goldman, 1968), (Pinus, 1968), (Vinnichenko, 1969), (Vinnichenko and Dutton, 1969), (Robinson, 1971) and indirectly by (Hwang, 1970)). For instance, on the basis of much more extensive measurements (Vinnichenko, 1969) commented that even if the mesoscale gap really existed, it could only be for less than 5% of the time; he then went on to note that Van der Hoven’s spectrum was actually the superposition of four spectra and that the extreme set of high frequency measurements were taken during a single one hour long period during an episode of ‘near-hurricane’ conditions and these were entirely responsible for the high frequency “bump”.

More modern temporal spectra are compatible with scaling from dissipation scales to \( \approx 10 \) days. Numerous wind and temperature spectra now exist from milliseconds to hours and days showing \( \beta \approx 1.6, 1.8 \) for \( v, T \) respectively, some of this evidence is reviewed in (Lovejoy and Schertzer, 2011). For longer periods fig. 4 shows daily station data analyzed up to 60 years with absolute slopes \( \beta = 1.6 \) indicated. This is roughly the same value of \( \beta \) as in the horizontal; it corresponds to an isotropic \( (x, y, t) \) space, a hypothesis to which we return below. According to these spectra, it is plausible that the scaling in the wind holds from dissipation scales out to scales of \( \approx 5 - 10 \) days where we see a transition (fig. 4). This transition is essentially the same as the low frequency “bump” observed by Van der Hoven; its appearance only differs because he used a \( \omega E(\omega) \) rather than \( \log E(\omega) \) plot.
Fig. 4: Wind spectra from 36 near complete 60 year series of daily data from the continental US taken from the stations lying nearest to 2°x2° degree grid points from 30°-50° N, 105° to 71° W. The brown curve (lower at left) is the daily maximum wind speed and the green is daily average (normalized so that the annual peaks coincide). The spectra have been also averaged in the frequency domain; in bins logarithmically spaced, 10 per order of magnitude (except for the lowest factor of 10 where no spectral averaging was performed). The reference lines have absolute slopes $\beta = 1$ (low frequencies), $\beta = 0.2$ (the plateau), and $\beta = 1.6$ (high frequencies).

The corresponding spectra for hourly surface temperatures, are shown in fig. 5, from daily series over 60 years long, fig. 6 (see also fig. 12 from daily series 6 years long). In all cases multiple series were averaged to reduce the noise. The reference lines have slopes corresponding (roughly) to the horizontally predicted values for the temperature (i.e. $\beta \approx 1.8$). Again, the horizontal values are seen to work well over the entire range up to the low frequency transition which in this case starts at around a 7 - 20 day scale (figs. 4, 5, 6, several figures in the next section and fig. 12).
Fig. 5: This shows the scaling of hourly surface temperatures from 4 stations in the northwest US, for 4 years (2005-2008) from the US Climate Reference Network. One can see that in spite of the strong diurnal cycle (and harmonics), that the basic scaling extends to about 7 days. The reference lines (with absolute slopes 0.2, 2 are theoretically motivated, see section 6. The spectra of hourly surface temperature data from 4 nearly colinear stations running north west - south east in the US (Lander, WY, Harrison NE, Whitman NE, Lincoln NE), from the US Climate Reference Network, 2005-2008. The thick line is the spectrum of the periodically detrended spectrum, averaged over logarithmically spaced bins.

Fig. 6: Mean spectrum of daily dew point temperature, $T_d$, temperature, $T$ and relative humidity, $h$ for 36, 33, 7 stations respectively, (numbers vary due to missing data) from stations with long (60 year; 22200 days) records. The low frequency and the “plateau” reference lines have slopes -1 and -0.2 respectively. The spectra were averaged over $1dB\omega$ bins (i.e. 10 per order of
magnitudes in frequency), every 2° from 30°-50° latitude, from -105° to -71° longitude. The high frequency reference line has absolute slope $\beta = 2$ close to the horizontal $\beta$ value for the humidity and temperature which are each about 1.8 - 1.9 (see table 1), and the plateau value is very close to the theory value 0.2. Adapted from (Lovejoy and Schertzer, 2010b).

| Exponent | Source | $h_s$ | $T$ | $u$ | $v$ | $w$ | $z$
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</table>

Table 1: Spatial and temporal Scaling exponents

These are from the ECMWF interim reanalysis and from various in situ estimates (for space, principally aircraft, (Lovejoy et al., 2010)). The temporal exponents were taken from the literature and when no other source was available, from daily station data that had been selected for a climate study; see the footnotes for details. Note that the ECMWF estimates were made at the (hyper) viscous dissipation scale whereas the in situ data were estimated in the scaling regime. Recalling the discussion in section 5, the difference is a factor $\approx 2.07$ for the velocity with possibly similar factors for the other variables. Due to the narrow range of frequencies (the Nyquist frequency (2 days)$^{-1}$) to about (5 days)$^{-1}$ before the spectra begin to flatten due to the transition to climate), direct estimates are not accurate and depend on the exact frequency range used. The zonal spatial outer cascade scales $L_{\text{eff}}$ were all in the range 13000 – 20000 km (except for $z$ which was $\approx 60000$ km), the meridional values were about a factor 1.6 smaller. The outer
time scales $\tau_{\text{eff}}$ were in the range 30-60 days with the exception of $z$ which was $\approx 300$ days. For full details, see (Lovejoy and Schertzer, 2011).

### 3.2 Temporal spectral scaling in the weather-climate regime

Excluding the annual cycle, roughly flat low frequency spectra are qualitatively reproduced in all the standard meteorological fields and the transition scale is relatively constant at typically 5-10 days. This regime was called the “spectral plateau” in (Lovejoy and Schertzer, 1986) to underline its ubiquity and to distinguish it from the higher frequency weather regime. In section 6, we shall see that the basic stochastic fractionally integrated flux turbulence model that reproduces the weather statistics when extended into the lower frequency predicts a dimensional transition leading to a spectral plateau with $\beta_{wc} \approx 0.2-0.4$ for essentially all the fields (at least those over land), over the ocean due to ocean turbulence it can yield $\beta_{wc} \approx 0.6$, hence in the figures discussed below we also systematically show theoretically predicted reference lines with these slopes. Finally, temporal spectra of the ECMWF 700 mb daily reanalysis data set for 2006 for $T$, $u$, $v$, $h_s$, $w$ and $z$ are shown in fig. 7. The overall shape and the transition scales are about the same as for the instrumental series, although the exponents are not necessarily the same; a detailed intercomparison is given in table 1.
Fig. 7: The temporal spectra of the daily 700 mb ECMWF interim reanalysis fields that were analyzed spatially in fig. 2 (i.e. between ±45° latitude). The dashed slopes are 5/3, the frequency spectrum is shown estimated using ensemble and spectral averaging, units of $\omega$: cycles/year. The dashed lines have slopes -5/3, the solid lines have slopes -3.35, -0.4, -2.4, -1.1 (top to bottom); they are drawn for $\omega > (11$ days)$^{-1}$. These correspond to the spatial $z$ exponent, the spatial $w$ exponent (which accurately fits $h_s$), the spatial $u, v, T$ exponent and the regression $w$ slope respectively. The curves top (on left) to bottom are $z, h_s$ (multiplied by 10 from spatial analysis, i.e. the spectra are multiplied by 100), $v, T, u$ and $w$ respectively. Note that the low frequency rise is due to only two frequencies $(1$ yr)$^{-1}$ and $(6$ months)$^{-1}$, it is a low resolution artefact of the annual cycle and subharmonic, not a true break in the scaling (see e.g. figs. 5, 8). Reproduced from (Lovejoy and Schertzer, 2010a).

Moving to lower frequencies, we can use the longest existing reanalysis, the 20th Century (20CR) reanalysis (Compo et al., 2009) from 1871-2008. A particularity of the 20CR is that it is based purely on surface pressure and monthly SST measurements. Here we use the 20CR data to compare different state variables at different locations, in particular at latitudes which are mostly continental or mostly maritime in character. Figures 8 a, b which show the spectra for $T, u, v, h_s$ from two 2° wide latitude bands, one tropical (centred on 5°N, mostly maritime), the other midlatitude centred on 45°N (mostly continental; both from the 20CR reanalysis product). We see that following the analysis of the temperature, that for the wind and humidity, the 45°N weather-climate plateau (fig. 8 a) follows the continental $\beta_{wc} \approx 0.2$ prediction very well up to around 10 years whereas the 5°N spectrum (fig. 8 b) is closer to the maritime value $\beta_{wc} \approx 0.6$ (with the notable exception of the meridional wind which displays a $\beta_{wc} \approx 0.2$ – 0.4 region as indicated). Overall, we conclude that all the data can be reasonably modelled by one or the other ($\beta_{wc} \approx 0.2$ or $\beta_{wc} \approx 0.6$) regimes being dominant.
Fig. 8 a: Spectra from the 20CR reanalysis (1871-2008) at 45°N. The reference lines have correspond to $\beta_{oc} = 0.6$, $\beta_{wc} = 0.2$, $\beta_w = 2$ left to right respectively.

Fig. 8 b: Spectra from the 20CR reanalysis (1871-2008) at 5°N. The reference lines correspond to $\beta_{oc} = 0.6$, $\beta_{wc} = 0.2$, $\beta_w = 2$ left to right respectively.

Our basic empirical findings are more or less in accord with a growing literature—particularly with respect to the temperature statistics. Our conclusions are especially close to those of (Huybers and Curry, 2006a) who studied many paleoclimate series as well as the 60 year long NCEP reanalyses and concluded that for periods of months up to about 50 years, the spectra are scaling with midlatitude $\beta_{wc}$'s larger than the tropical $\beta_{wc}$; (their values are 0.37±0.05, 0.56±0.08; our quite similar values of 0.2 – 0.4 and 0.6 work reasonably well, although as explained in section 4, 6 it seems that the appropriate distinction is between continental versus
maritime regions). Using observations and climate models at monthly resolutions, (Fraedrich and Blender, 2003), found $\beta_{wc} \approx 0$ over continents but $\beta_{wc} \approx 0.3$ in continental/ocean "transitional regions"; a similar finding was made by (Huybers and Curry, 2006a) who also noted a small mean land-sea difference (in their case of $\Delta \beta_{wc} = 0.2$); see table 2a.

(Bunde et al., 2004) gave evidence that over continents $\beta_{wc}$ was in the range 0.2 – 0.3 (similar to our estimates) and that the exponents implied the existence of long range correlations in accord with our continental results. The stations analyzed by (Bunde et al., 2004) (see also (Koscielny-Bunde et al., 1998) and (Lennartz and Bunde, 2009)) were primarily continental and their exponents were fitted from 3 months to 10 years so that they do not resolve the maritime versus continent issue ($\beta \approx 0.2 – 0.4$ versus $\beta \approx 0.6$). Recently, (Blender et al., 2006), argued that at least in some regions such as the North Atlantic (but not Pacific), there is indeed a long term memory (i.e. for Gaussian processes $\beta \neq 0$) which at these scales, they attribute to the thermal inertia of the Atlantic zonally averaged ocean circulation (see also (Fraedrich et al., 2009)).
<table>
<thead>
<tr>
<th>Series length (yrs)</th>
<th>$\beta_w$</th>
<th>$\tau_w$ (days)</th>
<th>$\beta_{wc}$</th>
<th>$\tau_c$ (yrs)</th>
<th>$\beta_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Northern hemisphere instrumental (Lovejoy and Schertzer, 1986), 1 month</td>
<td>100</td>
<td>–</td>
<td>–</td>
<td>$\approx 0.2^*$</td>
<td>3</td>
</tr>
<tr>
<td>20th C (6 hours) (700 mb), global</td>
<td>138</td>
<td>3</td>
<td>10</td>
<td>$\approx 0.6^*$</td>
<td>5</td>
</tr>
<tr>
<td>NOAA NCDC, NASA GISS, HadCRUT3 (surface, appendix 10.C)</td>
<td>129</td>
<td>–</td>
<td>–</td>
<td>0.2 (over land) 0.6 (SST)</td>
<td>10</td>
</tr>
<tr>
<td>Satellite global, (600 mb), daily</td>
<td>7</td>
<td>3</td>
<td>25</td>
<td>$\approx 0.2^*$</td>
<td>2</td>
</tr>
<tr>
<td>20thC (6 hours) (700 mb), 44N</td>
<td>138</td>
<td>2*</td>
<td>5</td>
<td>$\approx 0.2^*$</td>
<td>10</td>
</tr>
<tr>
<td>20thC (6 hours) (700 mb), 5N</td>
<td>138</td>
<td>2*</td>
<td>25</td>
<td>$\approx 0.6^*$</td>
<td>–</td>
</tr>
<tr>
<td>ECMWF interim (700 mb), year 2006</td>
<td>1</td>
<td>2*</td>
<td>5</td>
<td>$\approx 0.2, 0.6$</td>
<td>–</td>
</tr>
<tr>
<td>Instrumental, daily, US</td>
<td>60</td>
<td>2*</td>
<td>7</td>
<td>$\approx 0.2^*$</td>
<td>10</td>
</tr>
<tr>
<td>(Lanfredi et al., 2009); deviations &lt; 1 year considered as scale bound Markov process</td>
<td>$\approx 100$</td>
<td>–</td>
<td>–</td>
<td>0.26</td>
<td>–</td>
</tr>
<tr>
<td>(Huybers and Curry, 2006a) Midlatitude, tropics</td>
<td>Composite to $10^6$</td>
<td>–</td>
<td>–</td>
<td>0.4 (mid) 0.6 (tropics)</td>
<td>$\approx 0.1$</td>
</tr>
<tr>
<td>(Fraedrich and Blender, 2003)</td>
<td>NCEP reanalyses, 60yrs, ECHAM4/OPYC sims 500 yrs, in situ, 100 yrs</td>
<td>–</td>
<td>–</td>
<td>$\approx 0$ (continents), $\approx 0.3$ (coasts)</td>
<td>–</td>
</tr>
<tr>
<td>(Bunde et al., 2004; Koscielny-Bunde et al., 1998; Lennartz and Bunde, 2009),</td>
<td>In situ, 2 wks-30 yrs</td>
<td>–</td>
<td>14</td>
<td>0.2 - 0.3</td>
<td>–</td>
</tr>
<tr>
<td>(Ditlevsen et al., 1996)</td>
<td>Ice cores (GRIP)</td>
<td>–</td>
<td>–</td>
<td>$\approx 0.2$</td>
<td>3</td>
</tr>
<tr>
<td>(Lovejoy and Schertzer, 1986)</td>
<td>Composite ice cores, instrumental</td>
<td>1.8</td>
<td>25</td>
<td>$\approx 0$</td>
<td>3-300+</td>
</tr>
<tr>
<td>(Pelletier, 1998)</td>
<td>Composite: ice core, instrumental</td>
<td>1.5</td>
<td>7</td>
<td>0.5</td>
<td>200</td>
</tr>
</tbody>
</table>

Table 2a: An intercomparison of various estimates of spectral exponents $\beta$ and scaling range limits for the temperature.

The exponent $\beta_w$ is the high frequency weather regime exponent, $\tau_w$ is an estimate of the scale where the spectral plateau begins. $\beta_{wc}$ is and estimate of the exponent in weather-climate spectral plateau regime, and $\tau_c$ is a rough estimate of the long time (low frequency) limit to the latter which has exponent $\beta_c$.

*These values are not regression lines but from plausible reference lines indicated in the figures; in most cases, the value $\beta \approx 0.3$ would work nearly as well.

#Using monthly resolution GRIP ice core data for the last 3 kyrs.
The 3 year figure was using northern hemisphere instrumental series, the 300 yr figure was indirectly from Central England temperature series.

To complete this examination of the spectral plateau, let us consider a final relevant climate scale variables (see table 2 a for a summary of temperature series characteristics, see (Lovejoy and Schertzer, 2011) for further empirical analyses including of the North Atlantic Oscillation index). In our scaling model outlined in more detail in section 4.1, it is the balance of incoming and outgoing radiation (modulated by the clouds, abundances of H₂O, aerosols, CO₂, O₃, CH₄, N₂O (Tuck, 2008) and the surface variability all of which are scaling) that determine the overall energy flux (ε) that drives the weather. Therefore, we expect ε to be particularly significant for the climate. In fig. 9, we show an estimate from the 20CR reanalysis at 45°N based of the Laplacian of the (dominant) zonal wind component. We see that it is similar to the other 20CR fields although the plateau does indeed look very uniform and flat.

Fig. 9: The spectrum from 20CR reanalysis (1891-2002); the energy flux estimated from the absolute Laplacian of the zonal wind at 700 mb, 42°N. The reference lines have βc = 2, βwc = 0.2, βw =1. Note the daily and annual spikes and subharmonics.

<table>
<thead>
<tr>
<th>Series</th>
<th>Series length (yrs)</th>
<th>βw</th>
<th>τw (days)</th>
<th>βwc</th>
<th>τc (yrs)</th>
<th>βc</th>
</tr>
</thead>
<tbody>
<tr>
<td>20CR Reanalysis</td>
<td>T 5°</td>
<td>138</td>
<td>1.7</td>
<td>25</td>
<td>0.6</td>
<td>10</td>
</tr>
<tr>
<td>20CR Reanalysis</td>
<td>T 45°</td>
<td>138</td>
<td>3</td>
<td>4</td>
<td>0.2</td>
<td>10</td>
</tr>
<tr>
<td>20CR Reanalysis</td>
<td>u 5°</td>
<td>138</td>
<td>3</td>
<td>5</td>
<td>0.6</td>
<td>10</td>
</tr>
<tr>
<td>20CR Reanalysis</td>
<td>u 45°</td>
<td>138</td>
<td>2.6</td>
<td>4</td>
<td>0.2</td>
<td>10</td>
</tr>
<tr>
<td>20CR Reanalysis</td>
<td>v 5°</td>
<td>138</td>
<td>3</td>
<td>5</td>
<td>0.2</td>
<td>10</td>
</tr>
<tr>
<td>20CR Reanalysis</td>
<td>v 45°</td>
<td>138</td>
<td>3</td>
<td>4</td>
<td>0.2</td>
<td>10</td>
</tr>
</tbody>
</table>
Table 2b: An intercomparison of various spectral exponents and scaling range limits for different atmospheric variables.

<table>
<thead>
<tr>
<th>$h$, $5^\circ$</th>
<th>138</th>
<th>3</th>
<th>5</th>
<th>0.6</th>
<th>10</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$, $45^\circ$</td>
<td>138</td>
<td>3</td>
<td>4</td>
<td>0.2</td>
<td>10</td>
<td>0.6</td>
</tr>
<tr>
<td>US instrumental</td>
<td>$T$</td>
<td>60</td>
<td>2</td>
<td>7</td>
<td>0.2</td>
<td>3</td>
</tr>
<tr>
<td>$</td>
<td>\Psi</td>
<td>$</td>
<td>60</td>
<td>1.6</td>
<td>5</td>
<td>0.2</td>
</tr>
<tr>
<td>$p$</td>
<td>60</td>
<td>3</td>
<td>4</td>
<td>0.2</td>
<td>7</td>
<td>1.0</td>
</tr>
<tr>
<td>$R$</td>
<td>60</td>
<td>0.3</td>
<td>5</td>
<td>0.2</td>
<td>2</td>
<td>1.0</td>
</tr>
<tr>
<td>$h$, $R$</td>
<td>60</td>
<td>2</td>
<td>7</td>
<td>0.2</td>
<td>7</td>
<td>1.0</td>
</tr>
<tr>
<td>$T$ (hourly)</td>
<td>4</td>
<td>2</td>
<td>5-7</td>
<td>0.2</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Data from the 20CR reanalysis and daily stations over the US. The values are not regressions but reasonable reference lines, see the corresponding figures. The reanalysis resolution is 6 hrs, the US instrumental resolution is 1 day (except as indicated). Given the difficulty of estimating the exact limits of the scaling regimes and – for the low frequencies – the noisy statistics - the exponents $\beta_{wc}$, $\beta_{c}$ are not regression values but the exponents of reasonable reference lines chosen from 0.2 (the “plateau cascade” value for $\beta_{wc}$), 0.6 (the value of $\beta_{wc}$ claimed for the tropics, globe) and 1 (the value that roughly corresponds to the qualitative transition $H = 0$).

3.3 Temporal spectral scaling of global temperatures:

The 20CR data has several interesting characteristics including its coverage of a range of scales 6 hours to 138 yrs (a factor $> 2 \times 10^5$), with no missing data and whose characteristics are relatively homogeneous over time. However, it is based purely on surface pressure and monthly sea surface temperature data, thus raising the question of how representative is it of the real atmosphere. (Compo et al., 2009) give a partial answer by making specific comparisons with the NCEP reanalysis products, but systematic statistical studies have not yet been made. In order further answer to this question, we compare it to three surface temperature data sets.

The three we chose are the NOAA NCDC (National Climatic Data Center) merged land air and sea surface temperature dataset (abbreviated NOAA NCDC below, from 1880 on a $5^\circ \times 5^\circ$ grid, see (Smith, 2008) for details), the NASA GISS (Goddard Institute for Space Studies) data set (from 1880 on a $2^\circ \times 2^\circ$ (Hansen et al., 2010) and the HadCRUT3 data set (from 1850 to 2010 on a $5^\circ \times 5^\circ$ grid). HadCRUT3 is a merged product created out of the HadSST2 (Rayner et al., 2006) Sea Surface Temperature (SST) data set and its companion data set CRUTEM3 of atmospheric temperatures over land. The NOAA NCDC and NASA GISS are both heavily based on the Global Historical Climatology Network (Peterson and Vose, 1997), and have many similarities including the use of sophisticated statistical methods to smooth and reduce noise. In contrast, the HadCRUTM3 data is less processed.

Each pixel in each data set suffered from missing data points so that here we consider the globally averaged series obtained by averaging over all the available data over the common 129 yrs period 1880 – 2008 (taking into account the latitude dependent map factors; fig. 10 shows the spectra (see section 4.3 for analyses of the full $5^\circ$ resolution data). These are quite similar showing roughly $\beta_{wc} \approx 0.6$ up to $\approx 10$ yrs, and for $\omega < (10$ yrs)$^{-1}$, roughly $\beta_{c} = 1.7$, although clearly...
there is too much scatter at these low frequencies for more precise exponent estimates. Analysis
of the scale by scale differences between the spectra is interesting, see (Lovejoy and Schertzer,
2011). In terms of the real space statistics, we can use the with the global root mean square
\( \langle \Delta T(\Delta t)^{1/2} \rangle \) annual structure functions (fitted for 129 yrs > \( \Delta t > 10 \) yrs), obtaining
\[ \langle \Delta T(\Delta t)^{1/2} \rangle \approx 0.08 \Delta t^{0.33} \]
for the ensemble, in comparison, (Lovejoy and Schertzer, 1986) found the
very similar \( \langle \Delta T(\Delta t)^{1/2} \rangle \approx 0.077 \Delta t^{0.4} \) using northern hemisphere data (these correspond to \( \beta_c = 1.66, 1.8 \) respectively).

![Graph](image)

**Fig. 10**: The spectra (averaged over logarithmically spaced bins, 10 per order of magnitude, using the same colours as previous). The units are such that \( \omega = 1 \) corresponds to (129 yrs)\(^{-1} \); note the annual spike, (1 year\(^{-1} \)) is at 2.11 on the log\(_{10}\omega\) axis).

### 3.4 Temporal spectral scaling in the climate regime proper: 10 - 10^5 yrs

In the last few chapters we have attempted to use a scaling framework combined with
instrumental data to understand atmospheric variability over time scales ranging from
milliseconds to 10 – 100 yrs; the weather regime followed by a broad weather-climate plateau.
In order to properly characterize the longer scale variations (and even to have more confidence in
our instrumental statistics from decades to longer periods), we need to use paleo climate data.
Since the paleo data we consider here are primarily ice core temperature proxies, it turns out that
scales 10 years and less are close to their maximum resolutions so that there is not as much
overlap with instrumental series as one might wish. By combining the smaller paleo scales and
the largest instrumental scales we can obtain an overall “composite” view of the variability over
the full range, and this in turn will help understand the weather-climate plateau.

Thanks to several ambitious international projects, many ice cores exist, particularly in
the Greenland and Antarctic ice caps which provide surrogate temperatures based on \(^{18}\)O or
deuterium concentrations in the ice. The most famous cores are probably the GRIP (Greenland Ice core Project) and Vostok (antarctica) cores, each of which are over 3 km long (limited by the underlying bedrock) and go back respectively 240 and 420 kyr. Near the top of the cores, individual annual cycles can be discerned (in some cases going back over 10,000 yrs); below that the shearing of ice layers and diffusion between the increasingly thin annual layers makes such direct dating impossible, and models of the ice flow and compression are required. Various “markers” (such as dust layers from volcanic eruptions) are also used to help fix the core chronologies.

A problem with the surrogates is their highly variable temporal resolutions combined with strong depth dependences. For example, (Witt and Schumann, 2005) used wavelets, (Davidsen and Griffin, 2010) used (monofractal) fractional Brownian Motion as a model, and (Karimova et al., 2007) used (mono) fractal interpolation to attempt to handle this, (Lovejoy and Schertzer, 2011) found that the temporal resolution itself has multifractal intermittency. The main consequence is that the intermittency of the interpolated surrogates is a bit too high but that serious spectral biases are only present at scales of the order of the mean resolution or higher frequencies.

With these caveats table 3 summarizes some of the spectral scaling exponents, scaling ranges. It is interesting to note that the three main astronomical (“Milankovitch”) forcings at 19, 23 kyr (precessional) and 41 kyr (obliquity) are indeed visible – but only barely - above the scaling “background”. The predominance of the background is presumably associated with nonlinear variability within the climate system and was noted from the 1980’s onwards. The case for their relative dynamical insignificance was particularly strongly made by (Wunsch, 2003) who performed scaling analyses of various paleo spectra (see table 3), and who specifically attempted to determine the portion of the variance which could be accounted for by spectral bands near the three main Milankovitch frequencies. He concluded that even with a liberal account of these bands, that they represent no more than 10% of the total. Wunsch went on to propose various (nonscaling) Markov processes as models of the temperature.

<table>
<thead>
<tr>
<th>Series</th>
<th>Authors</th>
<th>Series length (kyrs)</th>
<th>Resolution (yrs)</th>
<th>$\beta_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Composite ice cores, instrumental</td>
<td>(Lovejoy and Schertzer, 1986)</td>
<td>Composite: Minutes to 10$^6$ years</td>
<td>1000</td>
<td>1.8</td>
</tr>
<tr>
<td>Composite (Vostok) (ice core, instrumental)</td>
<td>(Pelletier, 1998)</td>
<td>$10^{-5}$ to 1000</td>
<td>0.1 to 500</td>
<td>2</td>
</tr>
<tr>
<td>$\delta^{18}O$ from GRIP Greenland</td>
<td>(Wunsch, 2003)</td>
<td>100</td>
<td>100</td>
<td>1.8</td>
</tr>
<tr>
<td>Planktonic $\delta^{18}O$ ODP677 (Panama basin)</td>
<td>(Wunsch, 2003)</td>
<td>1000</td>
<td>300</td>
<td>2.3</td>
</tr>
<tr>
<td>CO$_2$, Vostok (Antarctica)</td>
<td>(Wunsch, 2003)</td>
<td>420</td>
<td>300</td>
<td>1.5</td>
</tr>
<tr>
<td>$\delta^{18}O$ from GRIP Greenland</td>
<td>(Ditlevsen et al., 1996)</td>
<td>91</td>
<td>5</td>
<td>1.6</td>
</tr>
<tr>
<td>$\delta^{18}O$ from GRIP Greenland</td>
<td>(Schmitt et al., 1995)</td>
<td>123</td>
<td>200</td>
<td>1.4</td>
</tr>
<tr>
<td>$\delta^{18}O$ from GISP Greenland</td>
<td>(Ashkenazy et al., 2003)</td>
<td>110</td>
<td>100</td>
<td>1.3</td>
</tr>
<tr>
<td>$\delta^{18}O$ from GRIP Greenland</td>
<td>(Ashkenazy et al., 2003)</td>
<td>225</td>
<td>100</td>
<td>1.4</td>
</tr>
<tr>
<td>$\delta^{18}O$ from Taylor (Antarctica)</td>
<td>(Ashkenazy et al., 2003)</td>
<td>103</td>
<td>100</td>
<td>1.8</td>
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<td>$\delta^{18}O$ from Vostok</td>
<td>(Ashkenazy et al., 2003)</td>
<td>420</td>
<td>100</td>
<td>2.1</td>
</tr>
<tr>
<td>Composite, Midlatitude</td>
<td>(Huybers and Curry, 2006a)</td>
<td>$10^{-4}$ to 1000</td>
<td>0.1 to 10$^4$</td>
<td>1.6</td>
</tr>
</tbody>
</table>
Table 3: An intercomparison of various estimates of the spectral exponents $\beta_c$ of the low frequency climate regime and scaling range limits.

For series with resolution $\approx 100$ yrs, the last three rows are for the (anomalous) Holocene only, see (Lovejoy and Schertzer, 2011).

### 3.5 A composite temporal spectrum from 6 hours to $10^5$ years

We have used a scaling framework and instrumental data in order to understand atmospheric variability over time scales ranging from milliseconds to over a century; the weather regime followed by a dimensional transition yielding a broad weather-climate plateau and then the beginning of a new low frequency climate regime proper. With the help of paleotemperature "surrogates", we have extended this picture out to the limits of quantitative climatology i.e. to $10^5-10^6$ yrs; we can now attempt to achieve a wide scale range composite of the overall variability.

The best known – and apparently the first – attempt to produce a wide scale range picture of atmospheric variability was (Mitchell, 1976). He produced what is still the most ambitious single composite spectrum of atmospheric variability to date: ranging from hours to the age of the earth ($\approx 10^4$ to $10^{10}$ yrs). Given the rudimentary quality of the data at that time, he admitted that his composite was more of an “educated guess” than a quantitative spectrum. We already mentioned that his model consisted of a juxtaposition of quasi-periodic processes (spectral “spikes”) with Ornstein-Uhlenbeck processes which in the spectrum would give rise to a series of “shelves”.

The first scaling composite model was (Lovejoy and Schertzer, 1986); which was based on some of the early paleo temperatures from Greenland and Vostok ice cores, an ocean core (although all were at the low resolutions available at that time) as well as local and hemispheric instrumental temperatures at scales of centuries and less. Key features were a) the distinction between the variability of regional and global scale temperatures b) finding that the global averages had particularly long scaling regimes, c) evidence for a scaling range for global averages between scales of about 3 years and 40 - 50 $kyrs$ (where the variability apparently “saturates”) with a climate exponent $\beta_c \approx 1.8$, d) the demonstration that this scaling regime could quantitatively explain the magnitudes of the temperature swings between interglacials.

A similar scaling framework was also adopted by (Pelletier, 1998) and resulted in a very similar composite; the main difference being the use of spectra rather than structure functions to make the composite. More recently, (Huybers and Curry, 2006a) made a more data intensive study of the scaling of many different types of paleotemperatures collectively spanning the range of about 1 month to nearly $10^6$ years. The result is qualitatively very similar to the previous
including the position of the scale breaks; the main innovations are a) the increased precision on
the β estimates and b) the basic distinction made between continental and oceanic spectra
including their exponents. We could also mention the composite of (Fraedrich et al., 2009)
which is a modest adaptation of that of (Mitchell, 1976) innovating by introducing a single
scaling regime spanning only two orders of magnitude: from \( \approx 3 \) to \( \approx 300 \) yrs (with \( \beta \approx 0.3 \)).

Fig. 11 shows our own updated composite where we have combined the 20CR reanalysis
spectra (both local, single pixel and hemispheric) with the GRIP 55 cm and GRIP high resolution
spectra. We could note that the use of the 700 mb reanalysis data is appropriate since this (rather
than the usual surface data) is roughly the level where the precipitation originates. We see that
the paleo and reanalysis spectra fit together very well. Since – unlike previous composites -
there is only a single “instrumental” source (the reanalysis) and single paleo calibration constant
for the GRIP \( \delta^{18}O \) series (which effectively allows all the paleo spectra to be simultaneously
moved up and down on the log-log plot), the results are relatively robust. From the figure we see
that whereas the local 75°N spectrum with \( \beta_{wc} \approx 0.2 \) overlaps well with the GRIP core out to its
limit \( \omega = (138 \text{ yrs})^{-1} \), that the spectrum of the northern hemisphere mean temperature instead
falls on the extrapolation of the climate regime part of the GRIP spectra (with \( \beta_{wc} \approx 1.4 \)). A
reference line with exponent \( \beta_{wc} \approx 0.6 \) - as suggested by the global scale and equatorial analyses
in fig. 8 – is also shown. This suggests a straightforward explanation for the difference between
the hemispheric and local temperatures: a climate process exists which varies the entire
hemispheric temperature and competes with the weather process. At weather scales, the climate
process has a very low amplitude, its variability (spectral power) is well below that of the
weather processes. However, since its exponent is much larger, at low enough frequencies it
becomes dominant. Starting at \( \omega \approx (10 \text{ yrs})^{-1} \) it first dominates the global scale average
temperature spectrum since this has a smaller amplitude; only later, (at around \( \omega \approx (300 \text{ yrs})^{-1} \))
does it dominate the more variable local temperature. For more scaling paleoclimate analyses,
including “paleo-cascades” (Lovejoy and Schertzer, 2011).

Fig. 11: A modern composite based purely on the GRIP core and 20CR reanalyses. All
spectra have been averaged over logarithmically spaced bins, 10 per order of magnitude. Spectra
of northern hemisphere temperatures (red; from the 20 CR reanalysis, 1871-2008), blue is the
single 2°x2°) pixel spectrum at 75°N also from the same reanalysis (and at the same scale). The light green is the mean of the GRIP high for last 90 kyrs and the (lowest) frequency blue is from the GRIP 55 cm core interpolated to 200 yr resolution and going back 240 kyrs. The solid reference lines have absolute slopes $\beta_{wc} = 0.2$, $\beta_{wc} = 0.6$ and $\beta_w = 2$ as indicated. The dashed line has absolute slopes $\beta_{wc} = 0.6$ which is suggested by the global scale and equatorial analyses in fig. 8. The red arrows at the bottom (and upper right) indicate the basic qualitatively different scaling regimes. Reproduced from (Lovejoy and Schertzer, 2011).

4. Anisotropic generalizations of the classical laws of turbulence and the spectral plateau

4.1 Real and Fourier Space-time scale functions

We have argued that atmospheric variables including the wind have wide range (anisotropic) scaling statistics and that spatial scaling of the horizontal wind leads to the temporal scaling of all the fields. Unfortunately, space-time scaling is somewhat more complicated than pure spatial scaling. At meteorological time scales this is because we must take into account the mean advection of structures and the Galilean invariance of the dynamics. At longer (climate) time scales, this is because we consider the statistics of many lifetimes (“eddy-turn-over times”) of structures. We first consider the shorter timescales. This is a summary of the more detailed discussion in (Lovejoy et al., 2008b), (Lovejoy and Schertzer, 2010b), (Lovejoy and Schertzer, 2011).

In order to illustrate the formalism, consider the horizontal wind $v$. In the stratified turbulence mentioned in section 2.3 (the 23/9D model), the energy flux $\varepsilon$ dominates the horizontal and the buoyancy variance flux $\phi$ dominates the vertical so that horizontal wind differences follow:

\[
\Delta v(\Delta x) = \varepsilon^{1/3} \Delta x^{1/3}; \quad a
\]
\[
\Delta v(\Delta y) = \varepsilon^{1/3} \Delta y^{1/3}; \quad b
\]
\[
\Delta v(\Delta z) = \phi^{1/5} \Delta z^{3/5}; \quad c
\]
\[
\Delta v(\Delta t) = \varepsilon^{1/2} \Delta t^{1/2}; \quad d
\]

where $\Delta x$, $\Delta y$, $\Delta z$, $\Delta t$ are the increments in horizontal coordinates, vertical coordinate and time respectively. Equations (3 a - b) describe the real space horizontal Kolmogorov scaling and 3 c the vertical Bolgiano-Obukhov (BO) scaling for the velocity, the equality signs should be understood in the sense that each side of the equation has the same scaling properties. The anisotropic Corrsin-Obukov law for passive scalar advection is obtained by the replacements $v \rightarrow \rho$; $\varepsilon \rightarrow \chi^{1/2} \varepsilon^{-1/2}$ where $\rho$ is the passive scalar density, $\chi$ is the passive scalar variance flux.

We have included eq. 3 d which is the result for the pure time evolution in the absence of an overall advection velocity; this is the classical Lagrangian version of the Kolmogorov law (Inoue, 1951; Landau and Lifschitz, 1959), it is essentially the result of dimensional analysis using $\varepsilon$ and $\Delta t$ rather than $\varepsilon$ and $\Delta x$. Although Lagrangian statistics are notoriously difficult to obtain empirically (see however (Seuront et al., 1996)), they are roughly known from experience
and are used as the basis for the space-time or “Stommel” diagrams that adorn introductory meteorology textbooks (see (Schertzer et al., 1997), (Lovejoy et al., 2000) for scaling adaptations).

The classical turbulence laws are isotropic whereas eq. 3a-d has different exponents in the horizontal ($x,y$), vertical ($z$) and time ($t$). However, the statistics implied by eq. 3 a-d have no characteristic lengths, they are anisotropic scaling generalizations of the classical laws. To see this explicitly, we can express the scaling (eqs. 3 a -d) in a single expression valid for any space-time vector displacement $\Delta R=(\Delta r,\Delta t)=(\Delta x,\Delta y,\Delta z,\Delta t)$ by introducing a scalar function of space-time vectors called the “(space-time) scale function”, denoted $\|\Delta R\|$, which satisfies the fundamental (functional) scale equation:

$$\left[\lambda^{-G_s}\Delta R\right]=\lambda^{-1}\|\Delta R\|; \quad G_s=\begin{pmatrix} G_s & 0 \\ 0 & H_t \end{pmatrix} ; \quad H_t=(1/3)/(1/2) = 2/3$$

(4)

where $G_s$ is the 3X3 matrix spatial generator:

$$G_s=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & H_z \end{pmatrix} ; \quad H_z=(1/3)/(3/5) = 5/9$$

(5)

(with rows corresponding to $(x,y,z)$) and $G_{st}$ is the extension to space-time. The spatial stratification model with $G_s$ given in eq. 5 has structure whose volume are proportional to $L,L,L^{H_z}=L^{D_{el}}$ with $D_{el}=2+H_z=23/9=2.55...$ $L$ is their horizontal extent, $D_{el}$ is the “elliptical dimension”; this is the “23/9D turbulence model” (Schertzer and Lovejoy, 1985b).

Using the space-time scale function, we may now write the space-time generalization of the Kolmogorov law (eq. 2) as:

$$\Delta \nu(\Delta R)=e^\eta_{\|\Delta R\|}H^{\|\Delta R\|}; \quad H = 1/3; \quad \eta = 1/3$$

(6)

where the subscripts on the flux indicate the space-time scale over which it is averaged. This anisotropic generalization of the Kolmogorov law is thus one of the key emergent laws of atmospheric dynamics and serves as a prototype for the emergent laws governing the other fields. Equation 6 is the prototypical emergent turbulent law, it relates fluctuations in an observable ($\Delta \nu$) to an underlying turbulent flux $\epsilon$. In section 5 we generalize it further by considering the cascade generation of highly intermittent (multifractal) fluxes. Note that although we do not consider it here, off diagonal elements of $G$ correspond to structures whose orientation as well as “squashing” depends on scale, further non matrix (nonlinear) Generalised Scale Invariance (Schertzer and Lovejoy, 1985a) is needed for structures whose morphologies change with location as well as scale (such as clouds, e.g. (Lovejoy and Schertzer, 2011)).

The simplest (“canonical”) space-time scale function satisfying eq. 4 is:

$$\|\Delta R\|_{can}=l_{s_s}\left(\frac{\|\Delta r\|}{l_s}\right)^2 + \left(\frac{\|\Delta t\|}{\tau_s}\right)^{2/H_z}$$

(7)
Where \( l_s = \phi^{-3/4} e^{5/4} \) is the dimensionally unique length scale called the “sphero-scale” and 
\( \tau_s = \phi^{-1/2} e^{1/2} \) is the corresponding “sphero-time” (the lifetime of structures of size equal to the sphero-scale). At scales \( l_s \), structures are typically roundish (since
\[ \{ (l_s,0,0,0) \} = \{ (0,l_s,0,0) \} = \{ (0,0,l_s,0) \} \], hence the name. With the scale function (eq. 7), the fluctuations (eq. 6) respect eqs 3 a- d.

Using the Lagrangian temporal scaling (eq. 3 d) implies \( H_s \neq 1 \) apparently predicting different horizontal and temporal scalings; this is in contradiction with the empirical analyses of the previous sections which showed that horizontal and temporal exponents were very close to each other. However, we are interested in the Eulerian frame temporal scaling for this, we are missing a key ingredient: advection. When studying laboratory turbulence generated by an imposed flow with velocity \( V \) with superposed turbulent fluctuations, (Taylor, 1938) proposed that the turbulence is “frozen” such that the pattern of turbulence blows past the measuring point sufficiently fast so that it doesn’t have time to evolve; i.e. he proposed that the spatial statistics could be obtained from time series by the deterministic transformation \( V \Delta t \rightarrow \Delta x \) where \( V \) is a constant: in the lab it is determined by the fan and by the wind tunnel geometry. While this transformation has been frequently been used in interpreting meteorological series, it can only be properly justified by assuming the existence of a scale separation between small and large scales so that the large scales really do blow the small scale (nearly “frozen”) structures past the observing point. Since we have argued that there is no scale separation in the atmosphere this is problematic.

As a first step in taking advection into account we can use the Gallilean transformation 
\[ \xi \rightarrow \xi - \xi t, \ t \rightarrow t. \] The second step is to consider \( \mathbf{v} \) as a random vector; with mean \( \left( \bar{v}_x, \bar{v}_y \right) \) (for simplicity we consider only the horizontal) and variance \( V^2 = \bar{v}_x^2 + \bar{v}_y^2 \) (in (Lovejoy and Schertzer, 2010b; Lovejoy and Schertzer, 2011) we give the full details). If the scaling holds to planetary scales, then this \( V \) is the typical velocity difference between structures at antipodes; 
\( V = V_w = e^{1/2} L_e^{1/3} \) where the scale of the earth is \( L_e = 20000 \) km. (the subscript “w” is for “weather”). From this, we can estimate the typical timescale \( \tau_w = L_e/V_w \) and use these to nondimensionalize the coordinates (denoted by a circumflex “\( \hat{\cdot} \)”). For more realism we can introduce the meridional/zonal aspect ratio \( a \) to obtain:

\[ \hat{\Delta}x = \frac{\Delta x}{L_w}; \quad \hat{\Delta}y = \frac{\Delta y}{a L_w}; \quad \hat{\Delta}t = \frac{\Delta t}{\tau_w}; \quad \hat{v}_x = \bar{v}_x \frac{\bar{v}_x}{V_w}; \quad \hat{v}_y = \mu_y \frac{\bar{v}_y}{V_w}; \quad (8) \]

the symbols \( \mu_x, \mu_y \) are used for the components of the nondimensional velocity; they are less cumbersome than \( \hat{v}_x, \hat{v}_y \) where:

\[ V_w = \left( \bar{v}_x^2 + a^2 \bar{v}_y^2 \right)^{1/2}; \quad \tau_w = \frac{L_w}{V_w}; \quad (9) \]

It is now convenient to define:
\[ \mu = (\mu_x, \mu_y) \quad |\mu|^2 = \mu_x^2 + \mu_y^2 \]  
\[ \Delta V(\overline{\Delta R}) = \mathcal{E}^{1/2} (\overline{\Delta R} B \overline{\Delta R})^{1/6} \]  
\[ \left\langle |\Delta I(\overline{\Delta R})|^q \right\rangle = \left\langle |\overline{\Delta R}|^{\xi(q)} \right\rangle ; \quad P_{st}(K) = \left\langle |\tilde{I}(K)|^2 \right\rangle = \left\langle |K| \right\rangle^{-s_{st}} ; \quad s_{st} = D_{st} - \xi(2) \]
Then we can use Kolmogov’s formula to extrapolate these first principles estimates up to a good job of explaining the horizontal wind fluctuations up to planetary scales.

\[ \frac{\Delta K}{\Delta R} = \left( \begin{pmatrix} \Delta R \\ \Delta T \end{pmatrix} \begin{pmatrix} B & \Delta R \end{pmatrix} \right)^{1/2} \; ; \; \frac{\Delta K}{\Delta R} = \left( \begin{pmatrix} \Delta K \\ \Delta T \end{pmatrix} \begin{pmatrix} B^{-1} & \Delta K \end{pmatrix} \right)^{1/2} \; ; \; \det B > 0 \]

These scale functions satisfy eq. 13; eq. 14 is the relation between their respective “trivial” anisotropies. In (Lovejoy and Schertzer, 2011) we shall see that the scale functions for non-space-time) localized (wavelike) behaviour \( \det B < 0 \). More details and an empirical verification of the above using geostationary MTSAT thermal IR data (including waves) is discussed in Pinel et al 2011, (Lovejoy and Schertzer, 2011).

**4.2 The weather-climate transition scale from “first principles”**

We have shown evidence that temporal scaling holds from small scales to a transition scale \( \tau_s \) of around 5 -10 days. Let us now consider the physical origin of this scale. In the original Van der Hoven \( \omega E(\omega) \) versus log \( \omega \) plot, it appeared as a low frequency spectral “bump” and its origin was argued to be due to “migratory pressure systems of synoptic weather map scale” (Van der Hoven, 1957). The corresponding features at around 4 – 20 days notably for temperature and pressure spectra were termed “synoptic maxima” by (Kolesnikov and Monin, 1965), and (Panofsky, 1969) in reference to the similar idea that it was associated with synoptic scale weather dynamics see (Monin and Yaglom, 1975) for some other early references.

If there is (at least statistically) a well defined relation between spatial scale and lifetimes (the “eddy-turn-over time”), then the lifetime of planetary scale structures is of fundamental importance. Since the shorter periods \( \tau < \tau_w \) statistics are dominated by structures of planetary size \( \tau = \varepsilon_{l}^{-1/3} l^{-2/3} < \tau_w \) whereas for \( \tau > \tau_w \), they are dominated by the statistics of many lifetimes of planetary scale structures. If we evaluate the lifetime at planetary scales \( L_e \) i.e. we assume that the outer weather scale \( \approx L_e \), and use eq. 3 a, b then we obtain a large scale velocity \( V_w \sim \varepsilon_w^{1/3} L_e^{1/3} \) which is the typical velocity across a structure of size \( L_e \). The corresponding eddy-turn over time/ lifetime of planetary structures is therefore \( \tau_w = L_e/V_w = \varepsilon_w^{-3/2} L_e^{2/3} \).

But what determines the globally averaged fundamental flux \( \varepsilon_w \)? We can estimate the mean energy flux \( \varepsilon_w \) by using the fact that the mean solar flux absorbed by the earth is \( \approx 200 W/m^2 \) (e.g. (Monin, 1972)). If we distribute this over the troposphere (thickness \( \approx 10^4 m \)), with mean air density \( \approx 0.75 Kg/m^3 \), and we assume a 2% conversion of energy into kinetic energy ((Palmén, 1959), (Monin, 1972)), then we obtain a value \( \varepsilon_w \approx 5 \times 10^{-4} m^2/s^3 \) which is indeed typical of the values measured in small scale turbulence (Brunt, 1939), (Monin, 1972). Using the ECMWF interim reanalysis to obtain a modern estimate of \( \varepsilon_w \) (Lovejoy and Schertzer, 2010b) showed that although \( \varepsilon \) is larger in mid latitudes than at the equator and that 300 \( mb \) it reaches a maximum, the global average is \( \approx 10^{-3} m^2/s^3 \). They also showed that using latitudinally varying \( \varepsilon \), that this explained to better than \( \pm 20\% \) the latitudinal variation of the hemispheric antipodes velocity differences (using \( \Delta v = \varepsilon^{1/3} L_e^{1/3} \)) and concluded that the solar energy flux does a good job of explaining the horizontal wind fluctuations up to planetary scales.

If we now assume that the horizontal dynamics are indeed dominated by the energy flux, then we can use Kolmogov’s formula to extrapolate these first principles estimates up to
Atmospheric and oceanic temperatures evolve with time and space scales, with planetary scales to estimate the large scale velocity difference across a hemisphere and we obtain $V_w \approx 17$ m/s. The corresponding eddy-turn over time, lifetime, is therefore $\tau_w \approx 10$ days i.e. roughly the time associated with synoptic – global scale phenomena, the transition to the spectral plateau as discussed in section 3.4.

Although this “first principles” calculation of the weather velocity $V_w$ and time scales $\tau_w$ from the solar energy input is seductive, as far as we can tell, it was not proposed until recently (Lovejoy and Schertzer, 2010b), presumably because the Kolmogorov law was believed to only hold in its isotropic form so that the classical relation $\tau \approx \epsilon^{-1/3} L^{2/3}$ couldn’t possibly apply to such large scales.

### 4.3 Ocean “weather”, the ocean - climate transition and the ocean spectral plateau

It is well known that ocean variability is important for atmospheric dynamics for months and longer time scales; before explicitly attempting to extend our model of weather variability beyond $\tau_w$, we must first consider the role of the ocean. The ocean and the atmosphere have many similarities; from the preceding discussion, we may expect analogous regimes of “ocean weather” to be followed by an ocean spectral plateau both of which will influence the atmosphere. To make this more plausible, recall that both the atmosphere and ocean are large Reynolds’ number turbulent systems and both are highly stratified - albeit due to somewhat different mechanisms. In particular there is no question that at least over some range that horizontal ocean current spectra are dominated by the ocean energy flux $\epsilon_o$ and hence follows roughly $E(\omega) \propto \omega^{-5/3}$ (and presumably in the horizontal: $E(k) \propto k^{-5/3}$); see e.g. (Grant et al., 1962), (Nakajima and Hayakawa, 1982). Although surprisingly few current spectra have been published, of late the use of satellite altimeter data to estimate sea surface height (a pressure proxy) has provided relevant empirical evidence and has somewhat revived the debate about the spectral exponent and scaling of the current. At scales where approximate geostrophic equilibrium may pertain, the pressure gradient is proportional to the current so that the surface height and current spectral exponents are related by $\beta_h = 2+\beta_w$. According to (Le Traon et al., 2008), at least over the scale range accurately covered by altimeters ($\approx 10$-300 km), $\beta_h \approx 11/3$ implying that $\beta_w \approx 5/3$ (in contradiction with the competing Quasi-Geostrophic prediction $\beta_w \approx 5$, (Smith and Vallis, 2001)); for more details and review, see (Lovejoy and Schertzer, 2011).

The existing data are apparently compatible with $k^{5/3}$ horizontal current spectra out to planetary scales, hence of the ocean energy flux $\epsilon_o$ is the dominant flux. Although empirically the current spectra (or their proxies) at scales larger than several hundred kilometres are not well known, other spectra, especially those of sea surface temperatures (SST) are known to be scaling over wide ranges and are relevant due to their strong nonlinear coupling with the current. Using mostly remotely sensed infra red radiances, and starting in the early 1970’s there is much evidence for SST scaling up to thousands of kilometres with $\beta \approx 1.8$ i.e. nearly the same as for the atmospheric temperature (see e.g. (McLeish, 1970), (Saunders, 1972), (Deschamps et al., 1981), (Deschamps et al., 1984), (Burgert and Hsieh, 1989), (Seuront et al., 1996), (Lovejoy et al., 2000), and a review in (Lovejoy and Schertzer, 2011)).
If, as in the atmosphere where the stratification is scaling, so it may well be in the ocean, leading to the likelihood that - as in the atmosphere - the energy flux dominates the horizontal ocean dynamics then we can use the same methodology as in the previous subsection - basic turbulence theory (the Kolmogorov law) combined with the mean ocean energy flux $\varepsilon_o$ - to predict the outer scale $\tau_o$ of the ocean regime. Thus, for ocean gyres and eddies of size $l$, we expect there to be a characteristic eddy turnover time (lifetime) $\tau = \varepsilon^{-1/3} l^{2/3}$ with a critical “ocean-weather” – “ocean-climate” transition time scale $\tau_o = \varepsilon_o^{-1/3} L_o^{2/3}$ ($L_o$ is the outer spatial scale of the oceans; presumably $L_o \approx L_e = 2 \times 10^7$ m). Again, we expect a fundamental difference in the statistics for fluctuations of duration $\tau < \tau_o$ - the ocean equivalent of “weather” with a turbulent spectrum with roughly $\beta_o \approx 5/3$ (at least for the current) - and for durations $\tau > \tau_o$, the ocean “climate” with a shallow ocean spectral plateau with $\beta \approx <1$.

In order to test this idea, we need $\varepsilon_o$, the globally averaged ocean current energy flux. As expected, $\varepsilon_o$ is highly intermittent (see (Robert, 1976), (Clayson and Kantha, 1999), (Moum et al., 1995), (Lien and D’Asaro, 2006), (Matsuno et al., 2006)) and as far as we know, the only attempt to estimate its global average is (Lovejoy and Schertzer, 2011) who argue that ocean drifter maps of eddy kinetic energy can be used to estimate it. They found that $\varepsilon_o \approx 10^{-8}$ m$^2$/s$^3$ is a reasonable global estimate for the surface layer (it decreases quite rapidly with depth). Using the formula $\tau_o = \varepsilon_o^{-1/3} L_o^{2/3}$ we find that a range of $\varepsilon_o$ between $1 \times 10^{-8}$ - $8 \times 10^{-8}$ m$^2$/s$^3$ corresponds closely to the range of $\tau_o \approx 1$ - 2 years; as expected, this is somewhat larger than the corresponding value for the atmosphere ($\varepsilon_o \approx 10^{-3}$ m$^2$/s$^3$, $\tau_o \approx 10$ days).

This provides us with a prediction for the SST spectrum: $E(\omega) \approx \omega^{1.8}$ for $\omega \approx (1 \text{ year})^{-1}$ and then a transition to a much flatter plateau for the lower frequencies. A way to test the model is to filter out high frequency weather variability due to the weather dynamics; this at least partially done by temporal averaging over scales $>\tau_o$. A one month resolution allows us to use the data sets described in section 3.4.

For all data sets and for virtually any given pixel, the series had many missing months, and these were often successive so that interpolation could easily lead to serious biases in the spectra and moments. To minimize this problem, we restricted the separate land and SST analyses to the most recent 100 years and selected only those pixels with less than 20 missing months. The mean spectra are shown in figure 12. While the land spectrum is – as expected - essentially a pure spectral plateau (with $\beta \approx 0.2$, the value cited earlier), we see that the SST spectrum is quite different displaying a clear transition between two power laws at $\tau_o \approx 1$ year (with $\beta \approx 0.6$, 1.8 for scales $>\tau_o$ and $<\tau_o$ respectively). Note also the rough convergence of the spectra at about 100 yr scale implies that the land and ocean variability become equal and the hint that there is a low frequency rise in the land spectrum for periods $>\approx 30$ yrs. Since above we predicted $\tau_o$ as about 1 year from estimates of $\varepsilon_o$, we see that the break in the empirical spectrum is very close to that predicted, although compared to the land temperature spectral plateau (representing more closely the free atmosphere), the low frequency ocean plateau $\beta$ is a little larger. In fig. 17, we will see that these conclusions are supported by direct evidence of planetary scale cascades with temporal outer scales $\approx 1$ yr as predicted.

The fact that both oceanic and atmospheric temperatures have high frequency $\beta$’s with the turbulent value $\approx 1.8$ support this interpretation, it also allows us to directly estimate the ratio
of atmospheric to oceanic energy fluxes $\varepsilon_w/\varepsilon_o$. This can be estimated from the left-right separation of the parallel $\omega^{1.8}$ lines which correspond to a factor of $\approx 30$ in critical time scales i.e. $\tau_o/\tau_w \approx 30$. If we assume that the spatial outer scale for the atmosphere and oceans is the same ($L_o = L_w$), then we can infer that $\varepsilon_w/\varepsilon_o = (\tau_o/\tau_w)^3 \approx 3 \times 10^4$. Using the atmospheric value $\varepsilon_w \approx 10^{-3} \text{ m}^2 \text{s}^{-3}$ we find $\varepsilon_o \approx 5 \times 10^{-8} \text{ m}^2 \text{s}^{-3}$ which is close to the estimates discussed above. If this interpretation is correct, then, since ocean eddies and gyres obey roughly the same turbulent phenomenology as the atmosphere, then the time scales $< \tau_o$ indeed correspond to ocean “weather”, with the implication that the limits to forecasting the ocean are $\approx \tau_o \approx 1 \text{ yr}$.

Fig. 12: This figure superposes the ocean and atmospheric plateaus showing their great similarity.

Left: A comparison of the monthly SST spectrum (bottom, blue) and monthly atmospheric temperatures over land (top, purple) for monthly temperature series from 1911-2010 on a $5^\circ \times 5^\circ$ grid; the NOAA NCDC data). Only those near complete series (missing less than 20 months out of 1200) were considered; 465 for the SST, 319 for the land series; the missing data were filled using interpolation. The reference slopes correspond to $\beta = 0.2$ (top), 0.6 bottom left and 1.8, bottom right. A transition at 1 year corresponds to a mean ocean $\varepsilon_o \approx 1 \times 10^{-8} \text{ m}^2 \text{s}^{-3}$.

Right: The average of 5 spectra from a sections 6 years long of a thirty year series from daily temperatures at a station in France (black, taken from (Lovejoy and Schertzer, 1986)). The red reference line has a slope 1.8 (there is also a faint slope 0 – flat - reference line). The relative up-down placement of this daily spectrum with the monthly spectra (corresponding to a constant
factor) was determined by aligning the atmospheric spectral plateaus (i.e. the black and purple spectra).

5. Cascades: generalizations of the classical turbulence laws for intermittency, multifractality

5.1 Intermittency and Cascades

Up until now, we have deliberately avoided discussing the statistics of the turbulent fluxes; we have only considered the generalization of the classical turbulence laws for anisotropy, but anisotropy was discussed at the level of second order statistics (power spectra). However, for realism we need another generalization of the classical laws to account for the intermittency or "spottiness" of turbulence (Batchelor and Townsend, 1949). Although at first – and even continuing today - cascades are often used in little more than a vague “conceptual” invocation, quantitative explicit cascade models were developed starting with (Novikov and Stewart, 1964), and by the 1980’s they were known to be the generic multifractal process.

There is now a large literature on cascades and multifractals, some of it was reviewed in (Schertzer et al., 1997), (Lovejoy and Schertzer, 2010b), (Lovejoy and Schertzer, 2011). The basic cascade idea is that starting at some external scale $L_{\text{eff}}$, nonlinear interactions iteratively break large structures into smaller ones with the “parents” multiplicatively modulating the “daughter” fluxes. For our present purposes, we need only consider a few cascade properties, notably that the general statistics of the cascades at resolution $\lambda'$ can be specified by their statistical moments via the simple multifractal cascade equation:

$$\langle \varphi^{q} \rangle = \lambda'^{K(q)}; \quad \lambda' = L_{\text{eff}} / L$$

where $\lambda'$ is the ratio of the “effective outer scale” $L_{\text{eff}}$ where the cascade begins and the resolution $L$ of the turbulent flux $\varphi$ and $K(q)$ is a convex function which specifies the statistics at all scales. In this chapter, we will be interested in empirical analyses in which the outer scale is not known a priori but is itself a significant empirically determined quantity. We will instead use the symbol $\lambda$ as the ratio of a convenient reference scale to the resolution; see below.

In general, cascades only have the weak convexity $K'' > 0$ and scale by scale conservation ($\langle \varphi^{q} \rangle = $ constant; $K(1) = 0$) constraints and are therefore impractical (otherwise we must deal with a nearly arbitrary unknown function ($K(q)$), the equivalent of an infinite number of parameters. Fortunately, due to a kind of multiplicative central limit theorem and under still rather general circumstances, (Schertzer and Lovejoy, 1987) and (Schertzer and Lovejoy, 1997) showed that cascades were expected to belong to the basin of attraction of “universal multifractals” in which only two parameters $C_1, \alpha$ are necessary:

$$K(q) = C_1 \left( q^{\alpha} - q \right) / (\alpha - 1)$$
The parameter $0 < C_1 < d$ (dimension of space) is the codimension of the mean, it characterizes the sparseness of the set of points giving the dominant contribution to the mean, $0 \leq \alpha \leq 2$ is the "multifractal index" which is the Levy index of the generator of the process; $\alpha = 0$ corresponds to a monofractal process ("$\beta$ model") and $\alpha = 2$ to a Gaussian generator ($\Gamma = \log \varepsilon$).

5.2 Estimating fluxes from the fluctuations

In order to test the general predictions of multiplicative cascades (eq. 15), we must analyze the data without relying on any specific theories of turbulence; we must use an approach that does not require a priori assumptions about the physical nature of the relevant fluxes; nor of their scale symmetries (isotropic or otherwise). If atmospheric dynamics are controlled by scale invariant turbulent cascades of various (scale by scale) conserved fluxes $\phi$ then in a scaling regime, the fluctuations $\Delta I(\Delta x)$ in an observable $I$ (e.g. wind, temperature or radiance) over a distance $\Delta x$ are related to the turbulent fluxes by a relation of the form:

$$\Delta I(\Delta x) \approx \phi \Delta x^H \quad (17)$$

(eq. 17 is a generalization of the Kolmogorov law for velocity fluctuations (the latter has $H = 1/3$ and $\phi = \varepsilon^3$, $\eta = 1/3$ where $\varepsilon$ is the energy flux to smaller scales). Without knowing $\eta$ nor $H$ - nor even the physical nature of the flux - we can use this to estimate the normalized (nondimensional) flux $\phi'$ at the smallest resolution ($\Delta x = l$) of our data:

$$\phi' = \phi / <\phi> = \Delta I / <\Delta I> \quad (18)$$

where "<->" indicates statistical averaging. Note that if the fluxes are realizations of pure multiplicative cascades then the normalized $\eta$ powers $\varepsilon^n / <\varepsilon>^n$ are also pure multiplicative cascades, so that $\phi' = \phi / <\phi>$ is a normalized cascade ($<\phi>$ is the ensemble mean large scale flux, i.e. the climatological value, it is independent of scale, hence there is no need for a subscript). The fluctuation, $\Delta I(\Delta x)$ can be estimated in various ways; in 1-D a convenient method (which works for the common situation where $0 \leq H \leq 1$) is to use absolute differences: $\Delta I(l) = |I(x + l) - I(x)|$ where $l$ is the smallest reliable resolution and where $x$ is a horizontal coordinate, (this is sometimes called "the poor man’s wavelet"; other wavelets could be used. In 2-D, convenient definitions of fluctuations are the (finite difference) Laplacian (estimated as the difference between the value at a grid point and the average of its neighbours), or the modulus of a finite difference estimate of the gradient vector. The resulting high resolution flux estimates can then be degraded (by averaging) to a lower resolution $L>l$.

Following eq. 15, the basic prediction of multiplicative cascades applied to a turbulent flux is that the normalized moments:

$$M_q = \langle \phi'^q \rangle = \langle \phi'^q \rangle / \langle \phi' \rangle^q \quad (19)$$

obey the generic multiscaling relation:

$$M_q = \lambda'^{K(q)} \left( \frac{\lambda}{\lambda_{eff}} \right)^{K(q)} ; \quad \lambda' = \lambda / \lambda_{eff} ; \quad \lambda = L_c / L ; \quad \lambda_{eff} = L_c / L_{eff} \quad (20)$$

where "<->" indicates statistical (ensemble) averaging and $L_{eff}$ is the effective outer scale of the cascade. $\lambda$ is a convenient scale ratio based on the largest great circle distance on the earth: $L_c = $
20,000 km and the scale ratio $\lambda/\lambda_{\text{eff}}$ is the overall ratio from the scale where the cascade started to
the resolution scale $L$, it is determined empirically, although from the foregoing discussion we
expect $L_{\text{eff}} \approx L_e$ so that we expect $\lambda_{\text{eff}} \approx 1$ corresponding to planetary scale cascades. Since even
at planetary scales each field nonlinearly interacts with the other fields, it is possible (and we
often find) that $L_{\text{eff}} > L_e$.

Let us now consider estimating the flux. There are two basic cases to consider. The first,
is widely applicable to empirical data which are nearly always sampled at scales much larger
than the dissipation scales, it is the one described above based on the scaling range formula, eq.
17. If instead we have dissipation range data (for example if we estimate fluxes from the outputs
of numerical models at the model dissipation scale), then the basic approach still works, but the
flux is generally not the same: for example in the case of the velocity field it is $\varepsilon^{1/2}$ (rather than
$\varepsilon^{1/3}$); see (Stolle et al., 2009).

5.3 An overview of spatial cascades

The spatial ECMWF cascade analyses are presented in fig. 13; for each variable we start
with the finite difference absolute Laplacian flux estimate which was then degraded by spatial
averaging in the corresponding direction and then statistical averaged over the other directions
(space and/or time). In the figures, one can clearly see the basic cascade structure of lines
converging to the external scales (this is predicted by eq. 15); note in particular that the external
cascade scales are systematically comparable to the largest great circle distance (20000 km), and
that the scaling is well respected at all but the largest scales (i.e. for $\log_{10} \lambda > 0.6$ i.e. for scales
$\approx 5000$ km: see below for error estimates). Here and below, references to “cascade structures”
are simply a convenient short-hand to indicate the converging straight lines predicted for the log
of the moments versus log of the scale for multiplicative cascades – it does not refer to real space
fluid structures. The moments are only shown up to order $q = 2$ since for large enough $q$ they
become dominated by the largest value present in the data sample so that the results spuriously
depend on the sample size $(K(q)$ becomes spuriously linear; this is a “multifractal phase
transition” see (Schertzer and Lovejoy, 1994)). It was found that in this data, the transition
always occurred for $q$ somewhat greater than 2, so that the moments shown here are well
estimated from the data.
Fig. 13: The analysis of the 700 mb fields at 0Z for 2006 between latitudes ±45°. The fluxes were estimated using finite difference Laplacians. The curves are the moments $q = 0, 0.1, 0.2, \ldots, 1.9, 2$ (top). $\lambda = 1$ corresponds the size of the earth, 20,000 km.

Very similar results were found for the forecast products of the Canadian Global Environment Model (GEM) at both $t = 0$ and $t = 144$, the National Weather Service Global Forecasting System model (GFS), and the 20CR reanalysis (see the review in (Lovejoy and Schertzer, 2011)). We find for example that the deviations are of the order ±0.3% for the reanalyses, ±0.3% for GEM and ±0.5% for GFS the 20CR reanalysis is nearly the same). These small deviations allow us to conclude that the analyses and models accurately have a cascade structure. Overall, from the table we can also see that the $K(q)$ “shape parameter” - the difficult to estimate multifractal index $\alpha$ - is roughly constant at $\alpha = 1.8\pm0.1$. 
Turning to in situ aircraft measurements (using the same 24 legs discussed in section 2.2) we refer the reader to fig. 14 which shows the flux analysis results for the temperature and humidity, potential temperature. (Lovejoy et al., 2010) analyzes this in detail including the wind field which is more complex due to its sensitivity to aircraft excursions in the vertical and concluding that as far as estimating horizontal scaling parameters is concerned, that the range 4 - 40 km is optimal (between the dashed lines in the figures): at smaller scales the trajectory is too intermittent while at the longer scales, one obtains isobaric rather than isoheight statistics. We nevertheless see fairly convincing cascade structure for the temperature, humidity and potential temperature. Once again, the outer scales are of the order of the size of the earth although the $L_{\text{eff}}$ for the wind is somewhat larger – the variability being presumably increased by the variability of the altitude (which – due to the aircraft response to turbulence - is also cascade-like with outer scale of 30 – 50 km see (Lovejoy et al., 2010)).

By using satellite radiance data (whose spectra were analyzed in fig. 1), (Lovejoy et al., 2009a) showed that the corresponding energy forcings and sinks (i.e. the short and long wave radiances) are also scaling with corresponding cascade structures – including estimates of their cascade parameters and outer scales. This is in accord with the spectral transfers analyzed in (Lovejoy and Schertzer, 2011) which did not have a clear direction. Other spatial cascades of remotely sensed data used the TRMM Precipitation Radar instrument (Lovejoy et al., 2008a) as well as the TRMM a passive microwave instrument (TMI; 5 wavelengths, 2 polarizations...
These TRMM results are bolstered by those from thermal infra red data from the geostationary satellite MTSAT ([Pinel, 2010]); see also (Harvey et al., 2002), (Gaonac’h et al., 2003). For reviews see (Lovejoy and Schertzer, 2010b) and (Lovejoy and Schertzer, 2011) and for topography cascades, relevant ot the lower atmospheric boundary condition, see (Gagnon et al., 2006).

5.4 An overview of the temporal cascade structure

We have already reviewed the evidence that atmospheric fields are scaling in time up to scale $\tau_w \approx 10$ days. Given the connection between space and time and evidence that the spatial cascade structure extends out to planetary scales as reviewed in the previous subsection, we should expect cascades up to time $\approx \tau_w$ although for the SST and for fields heavily influenced by the ocean, this may extend further to scales up to $\approx \tau_o \approx 1$ yr. In this section we review some of the evidence. Fig. 15 shows the first of these; the analysis of the ECMWF fluxes, the temporal analogues of the spatial analyses presented in fig. 13. Fig. 16 shows the temporal analysis of the 20CR reanalysis, in this case spanning 6 hours to 138 years. The parameters are summarized in table 1. In all cases, we see the same basic features: a cascade structure which is reasonably well respected up to scales of 5 -10 days with outer cascade scales typically in the range 20 – 60 days followed by a flattening at longer time scales. Note that the outer cascade scale $\tau_{eff}$ is a bit larger than $\tau_w$ which is the scale at which the scaling breaks down i.e. where the empirical curves diverge from the regression lines.
Fig. 15: The trace moments of the ECMWF interim reanalyses from daily data for 2006, the same as figs. 13 but for the temporal analyses, \( \lambda = 1 \) corresponds to 1 year. The effective outer temporal cascade scales (\( \tau_{\text{eff}} \)) are indicated with arrows. Reproduced from (Lovejoy and Schertzer, 2010a).
Fig. 16: The trace moments of the spatial Laplacians of Twentieth Century reanalysis products for the band 44°- 46°N for the zonal wind (upper left), meridional wind (upper right), the temperature (lower left) and specific humidity (lower right). The largest scale, $\lambda = 1$ corresponds to 138 yrs, the parameters of the fits are given in table 1. Notice the “bulge” in the $h_s$ moments up to scales of $\approx 1$ yr, possibly a reflection of the ocean cascade.

What about the predictions of section 4.3 that there is an ocean cascade to scale to $\approx 1$ yr? We have seen that the TRMM precipitation has an outer scale of this order but the other fields have outer scales closer to $\tau_w$. In order to see evidence of ocean cascades, we need to “filter out” the atmospheric variability, we can do this using the monthly data discussed in section 4.3 (fig. 12). Using the same SST and land data sets used for the spectral analysis, we can determine the cascade structures, see fig. 17. This not only shows that the spatial temperature scaling continues to near planetary scales, but also that the temporal cascades are almost identical for land and sea temperatures with accurate scaling to about 1 year and with outer cascade scales of about 3.5 years i.e. fully consistent with our estimate of $\tau_o$ above. To put this in perspective recall that the spectra, fig. 12 showed qualitatively and quantitatively different SST and air temperature over land spectra. Since the weather variability for $\tau_w$ has very low intermittency, the strong intermittency of these monthly averages is presumably almost entirely due to the ocean turbulence. This scaling intermittency due to turbulent ocean processes thus leads to small statistical deviations (even in the land statistics) from perfect power law scaling for periods less than a year. This was recently noted by (Lanfredi et al., 2009) who treated the deviations as a (scale bound) Markov process rather than a scaling one.
Fig. 17: This compares the cascade analysis of the SST (left column) and land series (right column) for space (zonal) top row ($\lambda = 1$ corresponds to $20000$ km), and time, bottom row ($\lambda = 1$ corresponds to 100 yrs) for the data discussed in fig. 14. The parameters and sampling details are given in section 3.3. It is particularly noteworthy that although the land and ocean cascade structures are nearly identical, that the corresponding spectra (fig. 19) are very different indicating that the intermittency of the land temperatures is controlled by the ocean “weather” variability. The longest “pure” land scale accessible was $\approx 165^0$ at midlatitudes hence the smallest accessible $\lambda \approx 10^{0.2}$ in the upper right graph.

6. Generalizing turbulent laws beyond $\tau_w$: the spectral plateau, Weather-Climate cascades and dimensional transitions

6.1 The fractionally Integrated Flux (FIF) model

We have shown that if the classical laws of turbulence are generalized to account for anisotropic scaling (eq. 6) and for intermittency (eq.15) that they are very accurately obeyed over huge range sof space and time scales and that in the time domain, they predict a scale break at time scales corresponding to the lifetimes of planetary size structures (at $\tau_w \approx 10$ days). In this section, we show how to further generalize the laws so as to reproduce the variability at time scales beyond $\tau_w$; indeed up to periods of the order of $10 – 100$ yrs. To make this generalization, we introduce the fractionally integrated flux (FIF) model (Schertzer and Lovejoy, 1987) which is an explicit dynamic stochastic model reproducing fields satisfying eq. 6, 15, 16.

Considering just the horizontal and time domains, we have argued that the velocity scale linking time and space should be precisely the typical velocity of the largest eddy $V$ which is determined by the external spatial scale $L$ (the size of the planet) and the driving global mean energy flux $\varepsilon_w$ (itself determined by the solar radiation modulated by the clouds and dynamics; $V_w = \varepsilon_w^{1/3}L_e^{1/3}$ and $\tau_w = L_e/V_w$. In other words, $L_e$, $V_w$, $\tau_w$ are determined from basic principles,
they are not simply adjustable model parameters. Considering the scale function for a planetary
scale region with small overall mean velocity (but not small rms velocity V), we argued that for
scales l < L and τ < τw, it was essentially isotropic (G_{eff} = the identity, trivial anisotropy only
given by eq. 12).
While we have spent effort testing the prediction of the basic model for scales l < Lc and τ
< τw, we have not examined the behaviour of the model for scales τ > τw. Can the same model
account for the weather–climate transition, and to what extent can it account for the climate
regime (e.g. the spectral plateau)? In other words, what are the limits of the model, at what
scales does it finally break down?
As a step toward answering this question, we turn to explicit space-time stochastic cascade
models. We first recall the basic features of (continuous in scale) cascade model for the
turbulent flux ε. First, since it is assumed to be a multiplicative process, it can be expressed in
terms of the exponential of an additive generator Γ:
\[ ε(r,t) = e^{Γ(r,t)} \]  \hspace{1cm} (21)
where Γ is the (dimensionless) generator and we have nondimensionalized ε by its ensemble
average. If we assume that the basic statistics are translationally invariant in space-time
(statistically homogeneous, statistically stationary), then Γ is given by a convolution between a
basic noise γ_{a}(r,t) (independent, identically distributed random variables), and g(r,t) is a Green’s
function (a deterministic weighting function that correlates them over (potentially) large space-
time distances):
\[ Γ(r,t) = γ_{a}(r,t) \ast g(r,t) \]  \hspace{1cm} (22)
For the stable and attractive processes leading to universal multifractals, γ_{a}(r,t) is taken as
a unit (and extremal) Levy noise, index α, i.e. whose second characteristic function is
\[ \log \langle e^{iqg} \rangle = q^{α}/(α - 1) \]. In addition for universal multifractals, g must have a particular form:
\[ g(r,t) = N_{D}C_{1}^{\frac{1}{α}}\Theta(t)\parallel(r,t)\parallel^{-D/α} \]  \hspace{1cm} (23)
with the singularity cutoff at the inner, dissipation scale and D the dimension of space-time (= the trace of the scale generator G for isotropic space-time), N_{D} is a normalization constant, C_{1} the
intermittency parameter of the mean intermittency, α is the Levy index of the (extremal)
uncorrelated space-time unit amplitude Levy noise γ_{a}(r,t) (see (Schertzer and Lovejoy, 1987) for
the basic model, (Marsan et al., 1996) for the extension to causal space-time processes, (Lovejoy
et al., 2008b) for the extension to turbulence driven waves, and (Lovejoy and Schertzer, 2009)
for a technical treatment of numerical issues. Causality has been taken into account with the use
of a Heaviside function Θ(t) (=0 for t<0, = 1 for t>0), (Marsan et al., 1996). Physically, the
noise γ_{a} represents the innovations and g the interaction strength.
The observable v (e.g. a horizontal wind component nondimensionalized by the outer
scale L_{c}) whose statistics obey eqs. 13 can be obtained from the flux by taking:
\[ v(r,t) = ε(r,t)^{1/3} \ast \left( Θ(t)\parallel(r,t)\parallel \right)^{-D/α} \]  \hspace{1cm} (24)
This is the “Fractionally Integrated Flux” (FIF) model, (Schertzer and Lovejoy, 1987).
6.2 From the weather to the climate: a dimensional transition to a new scaling regime characterized by H<0

We now consider the consequences of assuming that the FIF model holds for scales \( \tau \gg \tau_w \). In order to understand the basic features that the model predicts for the weather, the transition and the climate we can restrict our attention to a \( D = 3 \), \((x, y, t)\) section of the full \((x, y, z, t)\) model and ignore – for the moment - the complications associated with ocean augmented intermittency which are relevant over the regime \( \tau_w < \Delta t < \tau_o \). If we rewrite the equation for the cascade generator (eq. 22) nondimensionalizing \( r = (x, y) \) with \( L_c \) and \( t \) with \( \tau_w \), then we obtain for the generator \( \Gamma(x, t) = \log \varepsilon(x, t) \):

\[
\Gamma(r, t) = \int_{-\Lambda}^{\Lambda} \int_{-\Lambda'}^{\Lambda'} \gamma_a(r, r', t-t') g(r', t') dr' dt' + \int_{-\Lambda}^{\Lambda} \int_{-\Lambda'}^{\Lambda'} \gamma_a(r, r', t-t') g(r', t') dr' dt' \tag{25}
\]

\( \Lambda = L_c / L_t = \tau_w / \tau_i \) is the total range of meteorological scales \((L_i, \tau_i)\) are the inner, dissipation space and time scales) and \( \Lambda_c = \tau_c / \tau_w \) is ratio of the overall outer time scale of the climate process \( \tau_c \) to the outer time scale of the weather process; see fig. 18 for a schematic showing the ranges of integration in eq. 25. Eq. 25 is a convolution between the subgenerator noise \( \gamma \) which represents the “innovations” and the power law kernel \( g \) represents the interaction strength between scales physically and temporally separated by the space-time interval \((r', t')\). For \( \Lambda_c = \tau_c \gg \tau_w \gg 1 \) we therefore have approximately:

\[
\Gamma(x, t) \approx \Gamma_w(x, t) + \Gamma_c(t)
\]

\[
\Gamma_w(x, t) = \int_{-\Lambda}^{\Lambda} \int_{-\Lambda'}^{\Lambda'} \gamma_a(x-x', t-t') g(x', t') dx' dt'
\]

\[
\Gamma_c(t) = \int_{-\Lambda_c}^{\Lambda_c} \gamma_a(t-t') g(0, t') dt'
\]

where \( \gamma_a(t-t') \) is a spatially integrated Lévy noise. The region of integration in the weather integral \( S_\Lambda \) can be approximated by the half unit circle in \((x, y, t)\) space with the half circle around the origin of radius \( \Lambda_w^{-1} \) removed (half due to the Heaviside function, in fig. 18 only the positive quadrant is shown for clarity). The approximation in eq. 26 consists in assuming for \( \rho \gg \| \| \) that \( g(r, t) \approx g(0, t) \) so that for long enough time lags the spatial lags are unimportant. \( \Gamma_w(r, t) \) is a 3 D (space-time) integral corresponding to the contribution to the variability from the weather regime \((\Lambda_w^{-1} < r < 1, \Lambda_w^{-1} < \| t \| < 1)\), and the second \( \Gamma_c(r, t) \) is a 1-D (purely) temporal contribution due to the weather-climate regime. This drastic change of behaviour due to the change of space–time dimension over which the basic noise driving the system acts is a kind of “dimensional transition” between weather and climate processes. Figure 18 gives a schematic indicating that at small scales, the interactions occur over all spatial and temporal intervals (the interaction region is a space-time volume), whereas for long times, the interaction region is pencil-like, it is essentially 1-D. Physically this is a transition from the high frequency regime where both spatial
and temporal interactions are important, to a lower frequency regime where the dynamics are dominated by temporal interactions. In the former case, this means between neighbouring structures of all sizes and at their various stages of development whereas in the latter case, only between very large structures at various stages in their development.

Fig. 18: Schematic diagram showing the regions of integration in eq. 1 and the idea of “dimensional transition” when the region becomes pencil–like (1-D, large scales) rather than volume like (3-D, small scales), this is the dimensional transition. For clarity only the positive (x,y) quadrant is shown.

In this simplest model it is this separation into independent additive weather and climate generators with correlated noises integrated over spaces of different effective dimensions which is responsible for the statistical difference between weather and climate – plateau. At the level of the fluxes it means that the weather - climate process multiplicatively modulates the weather process at the larger time scales:

$$\varepsilon_{\Lambda_w,\Lambda_c}(t,x) = \varepsilon_{\Lambda_w}(x,t)\varepsilon_{\Lambda_c}(t)$$

with $\varepsilon_{\Lambda_w}(x,t)$, having the high frequency variability, $\varepsilon_{\Lambda_c}(t)$ the low frequency. The generic result is a “dimensional transition” in the form of a fairly realistic spectral plateau. Physically it means that whereas at small scales, there are significant dynamical interactions in space-time, at long time scales, the spatial interactions become unimportant.

Let us assume that the Green’s function $g$ is the same as discussed in section 4 for isotropic space-time so that for $D=d+1$ (spatial dimensions and time):
\[ g(r,t) = \Theta(t)\|r\|^D = \Theta(t)(|r|^2 + t^2)^{-\frac{(d+1)}{2\alpha}} \]

(i.e. taking \( B \) as the identity in eq. 11). In \( d \) spatial dimensions, from eq. 23, we see that the weather and weather-climate fluxes and generators are:

\[ \epsilon_w(t) = e^{\Gamma_w(t)}; \quad \Gamma_w = \left( \frac{C_1}{N_{d+1}} \right)^{1/\alpha} \int_1^\infty \frac{\gamma(t)dt}{t} \]

\[ \epsilon_{wc}(t) = e^{\Gamma_{wc}(t)}; \quad \Gamma_{wc} = \left( \frac{C_1}{N_{d+1}} \right)^{1/\alpha} \int_1^\infty \frac{\gamma(t)dt}{t} \]

The exponent in the integral for \( \Gamma_w \) has been chosen so that \( \epsilon_w \) be a true cascade process with \( \langle \epsilon_w^q \rangle = \lambda^{K(q)} \) and \( K(q) \) obeying eq. 16, however, the same choice in the equation for \( \Gamma_{wc} \) falls off more rapidly than for a \( d = 0 \) (pure temporal multifractal) cascade process (which requires \( t^{1/\alpha} \)), leading to low intermittency. The “plateau cascade” is called the “weather-climate” process because while the low frequency regime is new, it is ultimately predicted by the high frequency weather model; it is not a fully new climate regime.

The actual statistical behaviour of this regime is actually quite complex to analyze and has some surprising properties which are investigated in detail in (Lovejoy and Schertzer, 2011). The main characteristics are a) that although the bare process is still log-Levy, the weak correlations apparently lead to central limit convergence to Gaussians for the dressed statistics b) at large temporal lags \( \Delta t \) the autocorrelations \( \langle \epsilon_{wc}(t)\epsilon_{wc}(t-\Delta t) \rangle \) ultimately decay as \( \Delta t^{-1} \), although very large ranges of scale are necessary to observe it, c) since the spectrum is the Fourier transform of the autocorrelation and the transform of a pure \( \Delta t^{-1} \) function has a low frequency divergence, the actual spectrum of a finite range spectral plateau depends on the overall range of scales \( \Lambda_c \), d) over surprisingly wide ranges (factors of 100 – 1000 in frequency for realistic values of \( \Lambda_c \) in the range \( 2^{10} - 2^{16} \)), one finds “pseudo-scaling” with nearly constant spectral exponents \( \beta_{wc} \) which are typically in the range 0.2 – 0.4 for realistic values of \( \Lambda_c \), e) \( \beta_{wc} \) is independent of \( C_1 \) and only weakly dependent on \( \alpha \), f) the final fractional integration (eq. 24) for the observable has virtually no effect for \( \tau > \tau_w \) (the kernel has a strong high frequency divergence) so that the statistics are independent of \( H \).

The upshot of this is that we expect an overall scaling with \( \beta_{wc} \) whose value largely depends on the overall range of the plateau regime \( (\Lambda_c) \), pretty much independently of the values of \( \alpha, C_1 \) and \( H \). (Lovejoy and Schertzer, 2011) gives some of the details; for example, for fits over a range a factor 128 in scale, we obtain \( \beta_{wc} = 0.40, 0.33, 0.29, 0.23 \) with outer scales \( \tau_c \approx 30, 110, 450, 1800 \) yrs (with \( \alpha = 1.8, \tau_w = 10 \) yrs, i.e. corresponding to \( \Lambda_c = 2^{10}, 2^{12}, 2^{14}, 2^{16} \)). The (rough) empirical range of \( \beta_{wc} \approx 0.2 – 0.4 \) is thus compatible with \( \tau_c \approx > 30 \) yrs. In summary, we therefore find for the overall FIF model:
\[ E(k) \approx k^{-\beta_w}; \quad k > L_w^{-1} \]
\[ E(\omega) \approx \omega^{-\beta_w}; \quad \omega > \tau_w^{-1} \]
\[ E(\omega) \approx \omega^{-\beta_{wc}}; \quad \tau_c^{-1} < \omega < \tau_w^{-1} \]  

(30)

where \( \tau_c \) is the long external scale where the plateau ends (see discussion below) and the weather - climate spectral exponents are:

\[ \beta_w = 1 + 2H - K(2) \]
\[ 0.2 < \beta_{wc} < 0.4 \]  

(31)

As expected, the high frequency weather exponent is the usual one (with the usual structure function exponent \( \xi(q) = qH-K(q) \)), but the low frequency weather - climate exponent \( \beta_{wc} \) is new. Note in particular that it is independent of \( H \) and that \( \beta_{wc} > 0 \). As we just argued, in the weather-climate regime, the intermittency rapidly disappears as we “dress” the process by averaging over scales \( > \tau_w \) so that the weather-climate plateau is nearly monofractal and we have an effective climate exponent \( H_{wc} \):

\[ H_{wc} \approx -\frac{(1-\beta_{wc})}{2} \]  

(32)

so that (since \( \beta_{wc} < 1 \)), \( H_{wc} < 0 \) and the corresponding (generalized) structure function exponent at least approximately satisfies the monofractal linear relation:

\[ \xi_{wc}(q) \approx qH_{wc} \]  

(33)

Using \( 0.2 < \beta_{wc} < 0.4 \) corresponding to \( -0.4 < H_{wc} < -0.3 \), this result already explains the preponderance of spectral plateau \( \beta \)'s around that value already noted in many analyses presented in section 3. However, as we saw in fig. 12, the ocean plateau has a somewhat higher \( \beta_{oc} \approx 0.6 \) which implies \( H_{oc} \approx -0.2 \); in (Lovejoy and Schertzer, 2011) we use a simple coupled ocean - atmosphere model to show how this could arise as a consequence of double (atmosphere and ocean) dimensional transitions.

**6.3 Testing the FIF weather/climate model... or how to determine decadal scale climate variability from 1 Hz aircraft data**

In order to test the realism of the FIF model in reproducing the weather-climate plateau, we made a detailed comparison of temperature data and numerical simulations of the FIF weather model. The data were taken at 75° N (taken from the 20CR reanalysis, from 1871-2008). We averaged from 6 hourly to daily resolutions which resulted in a series 50404 days long for each longitudinal pixel. The simulation was of a simple \((x,t)\) cascade (see fig. 19) with the parameters: \( \alpha = 1.8 \), \( C_1 = 0.1 \), \( H_w = 0.5 \), \( \epsilon_w \approx 5 \times 10^{-4} m^2/s^3 \) (hence \( \tau_w = 10 \) days) observed by the NOAA Gulfstream 4 aircraft near 200 mb altitude; see table 1. The simulation was scaled so that its standard deviation coincided with that of the data. The latitude 75° N was chosen both
because it is largely dominated by land or ice covered ocean so that intermittency effects due to
the ocean circulation are not so strong, and because later we discuss and compare it with
Greenland paleotemperatures which are at about this latitude.

Except for the standard deviation and mean of the temperature at 75° N there was no
attempt whatsoever to “fit” the simulation to the reanalysis data so that we do not expect a
perfect data/simulation match. The object was to see how close the above “toy model” of a
dimensional transition can account for the atmospheric variability over large ranges of time
scales. In order to remove extraneous issues of sample size and series length, a 2^{17} x 2^4
simulation was made and the first 50404 spatial segments were taken (this is a bit less than half
the length and so avoids artificial correlations/ tendencies due to the periodicity of the
simulations).

Fig. 19: Top: 700 mb temperature data from the 20CR reanalysis (1871-2008) for 2°x2° pixels at
75N; the data was detrended and averaged over 16 days (1871 at left, 2008 at right); 16 pixels
spaced at 10° in longitude were used so as to exactly match the simulation. Below, we show a
time series of a single pixel of an (x,t) simulation 16x50404 pixels and then averaged over 16
simulated days; the effective scales of the simulation were 1 day in time and \( L_w/16 \approx 1200 \) km in
space, the parameters were \( \alpha = 1.8, C_1 = 0.1, H_w = 0.5 \) (close to the temperature as measured by
For graphical purposes, in addition to the standard deviation (= ±4.05 K), the simulation mean was adjusted to be the same as the data.

The comparison of the simulation and empirical spectra is shown in fig. 20. For clarity, we have averaged the spectrum over logarithmically spaced bins, ten per order of magnitude. Over the high frequency weather regime, the model and data agree quite well. This is not surprising since the aircraft data that were used to determine the parameters were at these smaller scales (recall that the overall standard deviations are the same; this determines the relative vertical placement of the curves in fig. 20). However what is not at all trivial is that the low frequency part of the spectrum - including the mean spectral exponent $\beta \approx 0.2$ - is also quite well reproduced; presumably the agreement would be better if the critical external scale was given a small adjustment. Certainly, ensemble averaging over many realizations of the FIF model give a fairly accurate slope $\beta = 0.2$ as indicated by the reference line. The high frequency aircraft parameters thus give a remarkably good model of the temperature spectra at 75°N out to decadal/centennial scales. In section 4.2 we saw that at somewhat larger scales, the climate regime proper emerges and the model is no longer accurate.

We should however mention that for more realism we should couple the ocean and atmosphere, simple models for this not only reproduce the cascade structure of monthly data (fig. 17), but also the otherwise anomalous exponent $\beta = 0.6$, further discussion of this model (including intermittency). For further model-data intercomparisons, see (Lovejoy and Schertzer, 2011).
7. Conclusions

Just the laws of continuum mechanics emerge from those of statistical mechanics when large enough numbers of particles are present, so do the laws of turbulence emerge at high enough Reynold’s numbers, strong enough nonlinearity. However, the classical turbulence laws were constrained by strong assumptions of homogeneity and isotropy and could not cover wide scale ranges in the atmosphere. By generalizing them to take into account anisotropy (especially vertical stratification) and intermittency, their range of applicability is vastly increased. In the last five years - thanks in part to the ready availability of huge global scale data sets of all kinds - it has been possible to verify that these generalized emergent laws accurately hold up to planetary scales, for a recent overview see (Lovejoy and Schertzer, 2010b).

These “weather regime” laws show that the horizontal variability is fundamentally controlled by solar forcing via the energy flux. First principles calculations show that this accurately accounts for the large scale winds and predicts a drastic “dimensional transition” at $\tau_w \approx 10$ days, the typical lifetime of planetary scale structures. Beyond this time scale, spatial interactions are rapidly quenched so that the longer scales are driven by essentially temporal degrees of freedom and the spectra of atmospheric fields display a shallow “spectral plateau”. We show that by making a third generalization of the classical laws that the behaviour in this “weather-climate” regime can be accurately predicted. The main complication is that - due to similar effects from ocean turbulence whose corresponding outer time scale $\tau_o \approx 1$ year – there is enhanced intermittency up to that scale, with a slightly steeper “ocean plateau” beyond. Depending on the field and location, the plateau continues up to scales of $\approx 10 – 100$ years beyond which the climate regime proper begins.

In this overview, we first review the basic evidence for wide range space-time scaling, first at weather scales, then at longer weather-climate (plateau) and climate scales using paleo-temperatures to quantify the variability up to scales of over 100 kyrs. We then review the basic theory for both anisotropic scaling (Generalized Scale Invariance) and for intermittency (multifractal cascades). Finally, we outline the third generalization to time scales $>\tau_w$, using the Fractionally Integrated Flux model. We give an example comparing numerical simulations with reanalysis data showing that the model works quite well. The overall picture is of three basic scaling ranges operating from milliseconds to $\approx 100$ kyrs, although we saw that the intermediate weather-climate regime is really an extension of the higher frequency atmospheric and “oceanic” weather. Rather than the climate being no more than long term weather, it thus emerges governed by new (scaling) laws. Weather and climate can thus be objectively distinguished by their different types of scaling variability, we can objectively answer the question “what is the climate”?

8. References


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**Figure Captions:**

Fig. 1: Spectra from $\approx 1000$ orbits of the VIRS instrument (Visible Infrared Scanner) on the TRMM satellite channels 1-5 (at wavelengths of 0.630, 1.60, 3.75, 10.8, 12.0 $\mu$m from top to bottom, displaced in the vertical for clarity). The data are for the period January through March 1998 and have nominal resolutions of 2.2 km. The straight regression lines have spectral exponents $\beta = 1.35, 1.29, 1.41, 1.47, 1.49$ respectively, close to the value $\beta = 1.53$ corresponding to the spectrum of passive scalars (= 5/3 minus intermittency corrections, see ch.3). The units are such that $k = 1$ is the wavenumber corresponding to the size of the planet (20000 km)$^{-1}$. Channels 1, 2 are reflected solar radiation so that only the 15600 km sections of orbits with maximum solar radiation were used. The high-wavenumber falloff is due to the finite resolution of the instruments. Adapted from (Lovejoy et al., 2008b).
Fig. 2: Inter-comparisons of the spectra of different atmospheric fields from the ECMWF interim reanalysis. Top (red) is the geopotential ($\beta = 3.35$), second from the top (green) is the zonal wind ($\beta = 2.40$), 3rd from the top (cyan) is the meridional wind ($\beta = 2.40$), 4th from the top (blue) is the temperature ($\beta = 2.40$) 5th from the top (orange) is the vertical wind ($\beta = 0.4$), at the bottom, (purple) is the specific humidity ($\beta = 1.6$). All are at 700 mb and between $\pm 45^\circ$ latitude, every day in 2006 at GMT. The scale at the far left corresponds to 20000 km in the east-west direction, at the far right to 660 km. Note that for these 2-D spectra, Gaussian white noise would yield $\beta = -1$ (a positive slope = +1).
Fig. 3: Aircraft spectra of temperature (blue, bottom), humidity (red, middle), log potential temperature (gold, top), reference lines $\beta = 2$. These are averages over 24 isobaric aircraft “legs” near 200 $mb$ taken over the Pacific Ocean during the Pacific Winter Storms 2004 experiment, the resolution was $280 m$ (Nyquist wavenumber = ($560 m$)$^{-1}$). Adapted from (Lovejoy et al., 2010).
Fig. 4: Wind spectra from 36 near complete 60 year series of daily data from the continental US
taken from the stations lying nearest to 2°x2° degree grid points from 30°-50° N, 105° to 71° W.
The brown curve (lower at left) is the daily maximum wind speed and the green is daily average
(normalized so that the annual peaks coincide). The spectra have been also averaged in the
frequency domain; in bins logarithmically spaced, 10 per order of magnitude (except for the
lowest factor of 10 where no spectral averaging was performed). The reference lines have
absolute slopes $\beta = 1$ (low frequencies), $\beta = 0.2$ (the plateau), and $\beta = 1.6$ (high frequencies).
Fig. 5: This shows the scaling of hourly surface temperatures from 4 stations in the northwest US, for 4 years (2005-2008) from the US Climate Reference Network. One can see that in spite of the strong diurnal cycle (and harmonics), that the basic scaling extends to about 7 days. The reference lines (with absolute slopes 0.2, 2 are theoretically motivated, see section 6. The spectra of hourly surface temperature data from 4 nearly colinear stations running north west - south east in the US (Lander, WY, Harrison NE, Whitman NE, Lincoln NE), from the US Climate Reference Network, 2005-2008. The thick line is the spectrum of the periodically detrended spectrum, averaged over logarithmically spaced bins.
Fig. 6: Mean spectrum of daily dew point temperature, $T_d$, temperature, $T$ and relative humidity, $h$ for 36, 33, 7 stations respectively, (numbers vary due to missing data) from stations with long (60 year; 22200 days) records. The low frequency and the “plateau” reference lines have slopes -1 and -0.2 respectively. The spectra were averaged over $1dB\omega$ bins (i.e. 10 per order of magnitude in frequency), every 2° from 30°-50° latitude, from -105° to -71° longitude. The high frequency reference line has absolute slope $\beta = 2$ close to the horizontal $\beta$ value for the humidity and temperature which are each about 1.8 - 1.9 (see table 1), and the plateau value is very close to the theory value 0.2. Adapted from (Lovejoy and Schertzer, 2010b).
Fig. 7: The temporal spectra of the daily 700 mb ECMWF interim reanalysis fields that were analyzed spatially in fig. 2 (i.e. between ±45° latitude). The dashed slopes are 5/3, the frequency spectrum is shown estimated using ensemble and spectral averaging, units of \( \omega \): cycles/year. The dashed lines have slopes -5/3, the solid lines have slopes -3.35, -0.4, -2.4, -1.1 (top to bottom); they are drawn for \( \omega > (11 \text{ days})^{-1} \). These correspond to the spatial \( z \) exponent, the spatial \( w \) exponent (which accurately fits \( h_s \)), the spatial \( u, v, T \) exponent and the regression \( w \) slope respectively. The curves top (on left) to bottom are \( z, h_s \) (multiplied by 10 from spatial analysis, i.e. the spectra are multiplied by 100), \( v, T, u \) and \( w \) respectively. Note that the low frequency rise is due to only two frequencies (1 yr)^{-1} and (6 months)^{-1}, it is a low resolution artefact of the annual cycle and subharmonic, not a true break in the scaling (see e.g. figs. 5, 8). Reproduced from (Lovejoy and Schertzer, 2010a).
Fig. 8 a: Spectra from the 20CR reanalysis (1871-2008) at 45°N. The reference lines have correspond to $\beta_\omega = 0.6$, $\beta_{w_c} = 0.2$, $\beta_w = 2$ left to right respectively.
Fig. 8 b: Spectra from the 20CR reanalysis (1871-2008) at 5°N. The reference lines correspond to $\beta_{oc} = 0.6$, $\beta_{wc} = 0.2$, $\beta_w = 2$ left to right respectively.
Fig. 9: The spectrum from 20CR reanalysis (1891-2002); the energy flux estimated from the absolute Laplacian of the zonal wind at 700 mb, 42°N. The reference lines have $\beta_c = 2$, $\beta_{wc} = 0.2$, $\beta_w = 1$. Note the daily and annual spikes and subharmonics.
Fig. 10: The spectra (averaged over logarithmically spaced bins, 10 per order of magnitude, using the same colours as previous). The units are such that $\omega = 1$ corresponds to $(129 \text{ yrs})^{-1}$; note the annual spike, $(1 \text{ year})^{-1}$ is at 2.11 on the $\log_{10}\omega$ axis).
Fig. 11: A modern composite based purely on the GRIP core and 20CR reanalyses. All spectra have been averaged over logarithmically spaced bins, 10 per order of magnitude. Spectra of northern hemisphere temperatures (red; from the 20 CR reanalysis, 1871-2008), blue is the single 2°x2° pixel spectrum at 75°N also from the same reanalysis (and at the same scale). The light green is the mean of the GRIP high for last 90 krys and the (lowest) frequency blue is from the GRIP 55 cm core interpolated to 200 yr resolution and going back 240 krys. The solid reference lines have absolute slopes $\beta_{wc} = 0.2$, $\beta_{wc} = 0.6$ and $\beta_c = 1.4$ and $\beta_w = 2$ as indicated. The dashed line has absolute slopes $\beta_{wc} = 0.6$ which is suggested by the global scale and equatorial analyses in fig. 8. The red arrows at the bottom (and upper right) indicate the basic qualitatively different scaling regimes. Reproduced from (Lovejoy and Schertzer, 2011).
Fig. 12: This figure superposes the ocean and atmospheric plateaus showing their great similarity.

Left: A comparison of the monthly SST spectrum (bottom, blue) and monthly atmospheric temperatures over land (top, purple) for monthly temperature series from 1911-2010 on a 5°x5° grid; the NOAA NCDC data, see table 5 for details). Only those near complete series (missing less than 20 months out of 1200) were considered; 465 for the SST, 319 for the land series; the missing data were filled using interpolation. The reference slopes correspond to $\beta = 0.2$ (top), 0.6 bottom left and 1.8, bottom right. A transition at 1 year corresponds to a mean ocean $\varepsilon_o \approx 1 \times 10^{-8} \text{m}^2\text{s}^{-3}$.

Right: The average of 5 spectra from a sections 6 years long of a thirty year series from daily temperatures at a station in France (black, taken from (Lovejoy and Schertzer, 1986)). The red reference line has a slope 1.8 (there is also a faint slope 0 – flat - reference line). The relative up-down placement of this daily spectrum with the monthly spectra (corresponding to a constant factor) was determined by aligning the atmospheric spectral plateaus (i.e. the black and purple spectra).
Fig. 13: The analysis of the 700 mb fields at 0Z for 2006 between latitudes ±45°. The fluxes were estimated using finite difference Laplacians. The curves are the moments $q = 0, 0.1, 0.2, \ldots, 1.9, 2$ (top). $\lambda = 1$ corresponds the size of the earth, 20,000 km.
Fig. 14: Temperature (top left), relative humidity (top right), log potential temperature (lower left right). From (Lovejoy et al., 2010).
Fig. 15: The trace moments of the ECMWF interim reanalyses from daily data for 2006, the same as figs. 13 but for the temporal analyses, $\lambda = 1$ corresponds to 1 year. The effective outer temporal cascade scales ($\tau_{eff}$) are indicated with arrows. Reproduced from (Lovejoy and Schertzer, 2010a).
Fig. 16: The trace moments of the spatial Laplacians of Twentieth Century reanalysis products for the band 44-46°N for the zonal wind (upper left), meridional wind (upper right), the temperature (lower left) and specific humidity (lower right). The largest scale, $\lambda=1$ corresponds to 138 yrs, the parameters of the fits are given in table 1. Notice the “bulge” in the $h_s$ moments up to scales of $\approx 1$ yr, possibly a reflection of the ocean cascade.
Fig. 17: This compares the cascade analysis of the SST (left column) and land series (right column) for space (zonal) top row ($\lambda = 1$ corresponds to 20000 km), and time, bottom row ($\lambda = 1$ corresponds to 100 yrs) for the data discussed in fig. 14. The parameters and sampling details are given in section 3.3. It is particularly noteworthy that although the land and ocean cascade structures are nearly identical, that the corresponding spectra (fig. 19) are very different indicating that the intermittency of the land temperatures is controlled by the ocean “weather” variability. The longest “pure” land scale accessible was $\approx 165^\circ$ at midlatitudes hence the smallest accessible $\lambda \approx 10^{0.2}$ in the upper right graph.
Fig. 18: Schematic diagram showing the regions of integration in eq. 1 and the idea of “dimensional transition” when the region becomes pencil-like (1-D, large scales) rather than volume like (3-D, small scales), this is the dimensional transition. For clarity only the positive (x,y) quadrant is shown.
Fig. 19: Top: 700 mb temperature data from the 20CR reanalysis (1871-2008) for 2°x2° pixels at 75N; the data was detrended and averaged over 16 days (1871 at left, 2008 at right); 16 pixels spaced at 10° in longitude were used so as to exactly match the simulation. Below, we show a time series of a single pixel of an (x,t) simulation 16x50404 pixels and then averaged over 16 simulated days; the effective scales of the simulation were 1 day in time and $L_w/16 \approx 1200$ km in space, the parameters were $\alpha = 1.8$, $C_1 = 0.1$, $H_w = 0.5$ (close to the temperature as measured by aircraft). For graphical purposes, in addition to the standard deviation (= ±4.05 K), the simulation mean was adjusted to be the same as the data.
Fig. 20: A comparison of the spectrum of the same data as in fig. 20 (bottom, blue points), with the simulation (top, red points); both spectra were averaged over logarithmic bins, 10 per order of magnitude. The reference lines have the theoretical slopes $\beta_w$, $\beta_{wc}$ indicated in eq. 31.