Scaling turbulent atmospheric stratification,

Part I: turbulence and waves

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In this first of a three part series, we critically re-examine theories of turbulence in a stratified atmosphere arguing that the dynamics should be buoyancy driven rather than energy driven and that they should be scaling, but anisotropic, not isotropic. We compare the leading statistical theories of atmospheric stratification which are conveniently distinguished by their elliptical dimension $D_s$ which quantifies their degree of spatial stratification. This includes the mainstream isotropic 2D ($D_s = 2$; large scales), isotropic 3D ($D_s = 3$; small scales) theory but also the more recent linear gravity wave theories ($D_s = 7/3$) and the classical fractionally integrated flux (FIF) $D_s = 23/9$ unified scaling model. In the latter, the horizontal wind has a $k^{-5/3}$ spectrum as a function of horizontal wavenumber determined by the energy flux and a $k^{-11/5}$ energy spectrum as a function of vertical wavenumber determined by the buoyancy force variance flux. Like the gravity wave models, the $23/9$D FIF model is scaling through the mesoscale, however, being based on turbulent fluxes it is more physically satisfying. The FIF model as originally proposed by Schertzer and Lovejoy in 1987 is actually a family of scaling models broadly compatible with turbulent phenomenology It is an anisotropic extension of the classical turbulent laws of Kolmogorov and Corrsin and Obukhov. However, until now it has mostly been developed on the basis of structures localized in space-time. In this paper, we show how to construct extreme FIF models with wave-like structures which are spatially localized but unlocalized in space-time, as well as a continuous family of intermediate models which are akin to Lumley-Shur models in which some part of the localized turbulent energy “leaks” into unlocalized waves.

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The key point is that the FIF requires two propagators (space-time Green’s functions) which can be somewhat different. The first determines the space-time structure of the cascade of turbulent fluxes, this must be localized in space-time in order to satisfy the usual turbulence phenomenology. In contrast, the second propagator relates the turbulent fluxes to the observables (wind, passiv scalar concentration etc.). Although the spatial part of the propagator is localized as before, in space-time it can be unlocalized (it is still localized in space, now in wave packets). We display numerical simulations which demonstrate the requisite (anisotropic, multifractal) statistical properties as well as wave-like phenomenologies. In parts II, III we examine the empirical evidence for respectively the spatial and temporal parts of the model using state-of-the-art lidar data of passive scalar and cirrus clouds.
1. INTRODUCTION

The modern view of the atmosphere is that of a turbulent hierarchy of interacting structures covering a wide range of scales. This picture largely originated in the pioneering work of Richardson (1922) on “Weather prediction by numerical process”. Although primarily concerned with laying the foundations of numerical forecasting, Richardson’s book also contains a footnote in the form of a poem where he proposes that the basic dynamic mechanism in the atmosphere is a cascade of eddies passing their energy from large to small scales, eventually undergoing viscous dissipation. Since the atmosphere has a very large aspect ratio (20,000 km/10 km; horizontal/vertical), we see that Richardson’s cascade picture immediately poses the question as to the nature of the stratification and how it should be incorporated into the cascade.

Since Richardson, fluid stratification has been modeled in several ways. Perhaps the most familiar is the dynamical meteorological approach which considers a homogeneously stratified atmosphere with constant Brunt-Väisälä frequency (i.e. with a constant gradient of potential temperature). Although this homogeneity assumption is a very unrealistic restriction on the dynamically significant range of scales, it is frequently used to give qualitative insight into the general effects of stratification, and it is sometimes used to interpret specific flows. In comparison, the “Boussinesq” approximation is less restrictive: it postulates the existence of a well defined “reference” vertical density profile $\rho(z)$ so that the buoyancy forces on a fluid particle are determined by the difference between the particle’s density and $\rho(z)$ rather than on the local difference in density with the surrounding fluid. This allows one to consider the small scales as isotropic fluctuations about an anisotropic large scale reference state. Other approaches attempt to only model the large scales by assuming from the outset that they are completely stratified (two dimensional, flat). The shallow water equations, the quasigeostrophic and barotropic approximations are commonly used in large scale models of this type.

These and other assumptions or approximations can be used as the basis for treating stratification in traditional turbulence approaches, i.e. statistical theories which attempt to consider the dynamics of structures spanning wide ranges of scale. Because the basic (e.g. Navier-Stokes) equations have no characteristic lengths (except the small millimetric dissipation scale, and the outer planetary scale), they classically admit isotropic scaling (“self-similar”) solutions. Turbulence approaches to atmospheric dynamics thus commonly break up the dynamically active range (planetary to dissipation scales) into subranges in which various scaling (power law) behaviors are supposedly dominant.

To date, the great majority of turbulence theories have postulated a priori that all the relevant regimes are isotropic. They thus require at least two regimes to model the atmosphere: a (quasi) two dimensional isotropic large scale and (quasi) three dimensional isotropic small scale. Since the scale height $H_s$ for the mean pressure is about 7.5 km, the “dimensional transition” from isotropic 2D to isotropic 3D turbulence must occur somewhere in the meso-scale; this is the origin of the elusive “meso-scale gap” in the energy spectrum which we discuss below. The main exceptions to isotropic scaling (“self-similar”) theories are the weakly nonlinear gravity wave theories (section 2.b.2) and the 23/9 D anisotropic “unified scaling” model proposed in (Schertzer and Lovejoy
Stochastic multifractal models following this scaling symmetry and obeying the multifractal extensions of the Kolmogorov and Corrsin-Obukhov statistics were proposed in (Schertzer and Lovejoy 1987); the Fractionally Integrated Flux (FIF) model. These involve the notion of scaling stratification and are the main focus of this three part paper.

To understand the meaning of scaling stratification, consider the dimension which characterizes the stratification. In a 2D atmosphere, there is only variability in the horizontal direction. In a 3D atmosphere, if $x$ and $z$ are horizontal and vertical and $\Delta \rho (\Delta x,0,0)$ designates density fluctuations in the horizontal and vertical directions over separation $\Delta x$, $\Delta z$ then if the variability is statistically isotropic - on average (“< >”) $\langle \Delta \rho (\Delta x,0,0) \rangle = \langle \Delta \rho (0,0,\Delta z) \rangle$. In an intermediate ($2<D<3$) stratified but still scaling case, we need only go a distance $(\Delta z/l_s) = (\Delta x/l_s)^{Hz}$ with $0<Hz<1$ to find that $\langle \Delta \rho (\Delta x,0,0) \rangle = \langle \Delta \rho (0,0,\Delta z) \rangle$. We see that $Hz=1$ corresponds to 3D isotropy and $Hz=0$ to 2D (isotropy in the horizontal plane). The dimensional parameter $l_s$ – the “sphero-scale” is the scale at which going a distance $\Delta x = \Delta z = l_s$ yields the same fluctuations: $\langle \Delta \rho (l_s,0,0) \rangle = \langle \Delta \rho (0,0,l_s) \rangle$. In scaling stratification, the aspect ratio of structures in vertical cross-sections therefore varies as a power law of scale; in addition, (assuming horizontal isotropy, i.e. $\Delta y = \Delta x$) the volume of typical structures varies as $\Delta x \Delta x \Delta x^{Hz}$ with $D_s = 2 + Hz$. The intermediate dimension $D_s$ is called an “elliptical dimension” because of the typical elliptical shapes of the vertical sections of the average structures. Note that the notion of dimension can also be used in a rather different sense to characterize the intermittency of this stratified turbulence. This different - multifractal - meaning is discussed in section 3 and parts II, III. As we shall see below, the proposal by (Schertzer & Lovejoy 1983b, 1985a, 1985b) that horizontal structures are dominated by energy fluxes, while vertical structures are dominated by buoyancy variance fluxes implies $Hz=5/9$ and hence $D_s=23/9$ (the “s” indicates “space”; below we consider the extension to space-time).

Since each of the above atmospheric models implies a specific elliptical dimension $D_s$ (or equivalently, $Hz$) it ought to be straightforward to empirically test them simply by measuring $D_s$ (or $Hz$) over the relevant ranges. The difficulty has been that until recently, tests have primarily been made using either aircraft wind data in the horizontal or balloon wind data in the vertical (the exception is (Lilley et al. 2004); lidar vertical cross-sections, see below and parts II, III). The results from separate experiments often from different parts of the world and under different conditions, can only be compared in an indirect way (with the partial exception of (Chigirinskaya et al. 1994), (Lazarev et al. 1994)). An additional problem is that aircraft do not fly in perfectly flat trajectories nor do balloons rise in perfect vertical paths so that in-situ wind velocity, temperature or density measurements made with such means can only be made over irregular trajectories. Indeed, it has only recently (Lovejoy et al. 2004) been discovered that – precisely due to non 2D turbulence - aircraft can follow fractal trajectories (i.e. can be biased with respect to linear trajectories over large distances) and analyses of drop sonde data (in progress) indicate that the same type of problem can affect in situ vertical soundings. Therefore, such in-situ data can yield spurious statistical exponents, spurious scaling breaks and erroneous interpretations. Finally, huge amounts of data are needed in order to average over the large fluctuations in order to obtain accurate results. These and
related empirical issues are the focus of part II, while part III extends this to the time domain.

In spite of several unsatisfactory theoretical assumptions - especially the coexistence of weakly nonlinear waves with strongly nonlinear turbulence driving - the gravity wave theories have the advantage of being close to the wave phenomenology commonly observed in the atmosphere and are roughly compatible with the empirical horizontal and vertical spectra. In contrast, the initial implementation of the main turbulence based anisotropic alternative – the Fractionally Integrated Flux (FIF) model (Schertzer & Lovejoy 1987), (Schertzer et al. 1997a) – which we will call “classical FIF” yields realizations clearly missing wavelike structures; Figs. 1(a) and (b) show a vertical section of lidar data (described in Part II), and a classical FIF simulation (with the empirically observed scaling and multifractal parameters). In spite of the similitude of the spectra in the horizontal, vertical and time (and higher order statistics as well), the classical FIF model is visually imperfect. However FIF is actually an extremely general scaling multifractal framework for turbulence dynamics, below, we show how to produce a continuum of models anywhere in between an extreme unlocalized (“waveline” FIF model to the classical localized one).

Figure 1(a). Vertical atmospheric section; lidar backscatter from a passive scalar (pollution; one of the data sets described and analyzed below), 3m resolution vertical, 96m horizontal $D_s = 2.55 \pm 0.02$ (numbers are pixels; there is a 32:1 aspect ratio). Note the wave-like undulations.
Figure 1(b). A simulated vertical cross-section of the 23/9D Fractionally Integrated Flux model for a passive scalar density (false colours), same aspect ratio as 1(a). This is the standard FIF model without wavelike fractional integration; it has the same sphero-scale and same differential anisotropy as the data in Fig. 1a, but lacks the waves. This and following models have same multifractal parameters statistically the same fluxes simulated with universal multifractals Levy index $\alpha=1.8$, codimension of the mean $=0.1$. For many more simulations (including in three spatial dimensions and with radiative transfer), see http://www.physics.mcgill.ca/~gang/Lovejoy.htm.

In outline, the spatial part of the FIF model is based on a physical scale function which defines the notion of scale in the stratified horizontal-vertical space. The idea is that the turbulent dynamics – energy and buoyancy variance fluxes - determine the dynamically appropriate notion of scale. From the physical scale function, the FIF then introduces two propagators (space-time Green’s functions); each a (generalized, anisotropic) power law; hence the term “Fractional Integration”. The first propagator determines the structure of the turbulent cascade, it must be localized in space-time in order to be compatible with the usual phenomenology of turbulent fluxes. The second propagator links the turbulent flux to the observable (e.g. velocity or passive scalar field). In the classical FIF, the two propagators were essentially the same. Below, we show how this classical FIF can be modified so that the second propagator – although based on the same physical scale function – can be wavelike – indeed it allows the classical stationary phase method of asymptotic approximation so that we can analyse the behaviour in the usual way in terms of wave packets, group velocities etc. Indeed, we show how specific gravito-turbulent dispersion relations can be chosen which are very close to the classical gravity wave dispersion relations. Our model thus leads to many predictions similar to the standard gravity wave theory yet is strongly nonlinear, requires no scale separation nor linear perturbation analysis (hence the waves and turbulence are inseparable, it is a “turbulence/wave” model). The overall result is statistically very close to the observations (including the strong intermittency).

The remainder of this paper is organized as follows. In Section 2, we review both isotropic and anisotropic theories of atmospheric turbulence. In Section 3, we review the classical FIF model and then outline the turbulence/wave model.
2. OVERVIEW OF STATISTICAL THEORIES OF ATMOSPHERIC STRATIFICATION

(a) The stably stratified atmosphere and isotropic theories of turbulence

(i) Discussion. We have already mentioned the dynamical meteorological approaches for understanding idealized flows; they first assume a homogeneously stratified atmosphere with a constant potential temperature gradient, they then consider the system’s stability when subjected to various vertical wind profiles/vertical shears. In spite of the unrealism of the homogeneity assumption, this model is frequently used to give qualitative insight into the general effects of stratification. In comparison, turbulence theories (and Direct Numerical Simulations) generally use the Boussinesq approximation. This starts by postulating a scale separation between the large stratified scales and the small turbulent scales (which are usually considered as isotropic perturbations). The scale separation allows a vertical density profile to be defined and fluid particles are hypothesized to experience a buoyancy force determined by the difference between their density and that of the vertical average density rather than of their local environment. In numerical modeling, variants of the Boussinesq approximation which share its basic scale separation hypothesis are the hydrostatic approximation or the aenelastic approximation. Still other approaches essentially just treat the large scales and consider the system as 2D either via the shallow water equations or the quasi-geostrophic and barotropic approximations. The small scales are usually thought of as three-dimensional isotropic turbulent perturbations around a large scale anisotropic state.

(ii) Dynamically driven 3D turbulence. The theoretical paradigm of 3D isotropic turbulence was originally developed as a way of simplifying the problem of fully developed dynamically forced turbulence and was initially applied to laboratory isotropic grid-turbulence. The key result is the famous Kolmogorov 1941 $k^{-5/3}$ spectrum for the velocity variance - in incompressible flow, this equals the kinetic energy per unit mass - between wavenumbers $k$ and $k+dk$ with spectral exponent $\beta=5/3$. This result is believed to follow from three symmetries of the Navier-Stokes equations: (a) their scaling symmetry, (b) the conservation of energy flux from large scales to small scales by the nonlinear terms, (c) the Fourier space “locality” of the nonlinear interactions. We use adjective “believed” because there is still no formal mathematical proof and at very least intermittency corrections are necessary. Property (a) is a formal property of the Navier-Stokes equations: the symmetry is broken at large scale by the energy injection and at small scales by the viscous dissipation but is held to be approximately valid over the wide intermediate range. Property (b) explains why it is the energy flux which is the cascading quantity, while property (c) implies that large structures don’t spontaneously break up into many much smaller ones, that the transfer of energy flux - conserved according to (b) – mostly occurs via the gradual break-up of structures into slightly smaller ones. Together, the three properties form the basis of cascade models discussed in Section 3. Application to turbulent flows usually involve the assumption of a quasi-steady flux of energy from large scales to small scales. In other words, although the turbulent system will generally be far from thermodynamic equilibrium, there can nevertheless be a quasi-
steady state in which the large scale forcing is on average balanced by small scale dissipation. With this assumption, the Kolomogorov spectrum is a kind of “mean field” result obtained largely by dimensional analysis, ignoring the effects of spatial variability, intermittency. Statistical mechanical type arguments can be used to show that in 3D, the direction of the energy transfer is primarily from large to small scale, i.e. the cascade is downscale (also called a “direct” cascade). In spite of over sixty years of effort - and the acknowledgement that flows from the Earth’s atmosphere and ocean to Jupiter’s atmosphere through to intergalactic jets follow $k^{-5/3}$ at least over some of their range - there is still no analytical derivation of $k^{-5/3}$ from the governing equations.

(iii) 2D dynamically driven turbulence. Since large scale structures cannot be modelled by three dimensional isotropic turbulence, alternatives are needed. Starting with (Fjortoft 1953), both (Kraichnan 1967) and (Leith 1968) considered turbulence of a purely two dimensional fluid. This case is rather special since in 2D, the vortices are all vertically aligned, so that the main mechanism for three dimensional turbulence – vortex stretching – is prohibited. The result is that there exists, in addition to the energy flux ($\varepsilon$), another quadratic invariant, the mean square vorticity flux: the “enstrophy”. Repeating Kolmogorov-type arguments for this dimensional quantity leads to a $k^{-3}$ direct enstrophy cascade regime towards small scales while at large scales there is an inverse (smaller to larger scales) cascade of energy flux. (Note that when necessary we will specify a horizontal wavenumber by $k_x$, and a vertical one by $k_z$). The application of 2D turbulence to the atmosphere requires a number of assumptions about energy and enstrophy injection scales and mechanisms, as well as the specification of the role of three dimensional turbulence; see section 2.1.5. A consequence of the much steeper spectral slope $\beta_x=3$ is that two-dimensional turbulence is borderline “nonlocal”. This is associated with a number of peculiarities such as sensitivity of the statistics to boundary conditions; numerical simulations often find spectra closer to $k^{-4}$ (see e.g. (Kida & Yamada 1984), (Brachet et al. 1988)).

The main application of 2D turbulence to the atmosphere was the notion of “quasi geostrophic potential vorticity” which (Charney 1971) showed leads to 2D turbulence in a barotropic atmosphere. In any case, in spite of several claims of $k^{-3}$ horizontal behaviour (but always over very short ranges e.g. (Nastrom et al. 1984)), there is no clear evidence for 2D turbulence in the atmosphere (see part II).

(iv) Buoyancy driven turbulence. Fundamentally the atmosphere is buoyancy-driven via incoming solar radiation, hence a theory postulating a driving mechanism based on buoyancy forces is more appropriate to the atmosphere than one simply based on an unspecified large scale stirring/forcing mechanism. Bolgiano (1959) and Obukhov (1959) proposed such a buoyancy-driven mechanism using the Boussinesq approximation although unfortunately, they assumed it to be isotropic. They found that the temperature variance flux (in fact proportional to a buoyant energy flux) obeys the equations of passive scalar advection, and that there exists a new quadratic invariant, a new dimensional quantity $\phi$, with dimensions length$^2$/time$^5$. Their original proposal was for a “buoyancy subrange” with an isotropic $k^{11/5}$ spectrum, which, according to the Boussinesq approximation would dominate the energy flux (i.e. Kolmogorov) subrange for scales larger than the Bolgiano scale $l_B$ (see below) itself estimated to be of the order of meters. In the following years, this theory was largely discarded because of the failure
to empirically detect this isotropic range either in time or in the horizontal direction. By the time the $k_z^{-11/5}$ spectrum was observed in horizontal wind (Endlich 1969; Adelfang 1971) it was almost forgotten. It wasn’t until the 1980’s that the basic idea was revived in an anisotropic form as part of the 23/9 D model.

(v) *Isotropic gravity wave dynamics.* If one accepts that buoyancy forces play a significant role, there are two significant quadratic invariants $(\varepsilon, \phi)$ each depending only on dimensions of space and time, hence dimensional analysis no longer provides a unique spectral exponent. A new physical model is therefore required. Originally motivated by observations of the vertical velocity with either a $k_x^{-5/3}$ or $k_x^{-3}$ spectrum, rather than $k_x^{-11/5}$, (Shur 1962) and (Lumley 1964) postulated that there exists an isotropic Kolmogorov-type regime in which the basic dynamics are provided by energy fluxes but also proposed a leakage mechanism in which the kinetic energy flux is reduced by buoyancy effects over the entire inertial range. This causes continuous variation of $\varepsilon$ throughout the range. The leakage idea was later refined by (Weinstock 1978). Note that the Lumley-Shur theory of the buoyancy subrange predicts a spectral density function that does not explicitly require the underlying motions to obey gravity wave dispersion relations. Nevertheless, the overall effect of the energy leakage is to produce a more or less isotropic large scale $k^{-3}$ range which smoothly interpolates to an isotropic $k_x^{-5/3}$ spectrum at smaller scales with a transition occurring at the Bolgiano scale. It should be noted however the use of the Boussinesq approximation in this theory limits the role of buoyancy to small perturbations, and that $k^{-3}$ has not been observed in the horizontal anywhere near the theoretical Bolgiano scale.

(vi) *Attempts to combine several isotropic regimes.* The insistence on considering only isotropic models of turbulence lead to what might be called the “standard model” of the atmosphere. Starting with the “meso-scale gap” (Panofsky & Van der Hoven 1955; Van der Hoven 1957) separating the large and small scales, and followed by a direct (downscale) $k^3$ enstrophy cascade (Kraichnan 1967), the small scales were increasingly regarded as 3D turbulent perturbations on the large scale (2D turbulence) “weather”. This picture was consecrated in the influential book by (Monin 1972) - but not without difficulty. An outstanding one being that in 2D turbulence, the energy flux cascades to larger scales (i.e. “inversely” where there are no obvious sinks to prevent huge energy build-ups. In addition, there is consensus on the observations of $k_x^{-5/3}$ spectra from small scales to horizontal scales much larger than the atmospheric scale height 10 km. For isotropic models, this implies a complex hierarchy of cascades in which the small scales are characterized by 3D isotropic downscale energy cascades, the meso-scales by a downscale enstrophy cascade, and the larger scales by an upscale energy cascade. In order to work, each must have appropriate energy (and for the 2D regime, enstrophy) sources and sinks (see e.g. (Gage 1979; Nastrom & Gage 1983; Lilly 1983) for proposals in this direction). Additional theoretical difficulties were raised by (Bartello 1995), (Lilly et al. 1998) and (Riley & Lelong 2000) who pointed out the awkward possibility that the small scale three dimensional regime could destabilize the quasi geostrophic 2D regime.

(b) *Gravity wave based anisotropic theories*

(i) *The validity of the Boussinesq approximation and unstable stratification.* A further difficulty faced by isotropic subrange models of the stratified atmosphere is that a priori gravity acts at all scales so that theoretically one does not expect there to be scale
separations of the kind needed to justify Boussinesq-type approximations. The usual justification of these approximations is empirical; it is argued that when one considers thick enough layers, the atmosphere usually seems to be stably stratified. This conclusion is based on two arguments. The first is the classical analysis of the terms of the governing equations: if $\theta$ is a reference potential temperature and $\Delta \theta$ a perturbation, one can argue that we require $\Delta \theta/\theta = \Delta \log \theta \ll 1$ for the application of the Boussinesq approximation (see e.g. (Lesieur 1987)). A somewhat more precise argument involves the static stability

$$B = \frac{d \log \theta}{dz}$$

of an infinitesimal fluctuation $d \log \theta$, which is related to the Brunt Väisälä frequency $N$ via

$$N = (gB)^{1/2}$$

where $g$ is the acceleration of gravity. For the Boussinesq approximation to be valid, we require $B$ to be less than the inverse thermodynamic scale height $H_s = 7.5$ km (see e.g. (Green 1999)). In the usual treatment, “typical” order of magnitude estimates for atmospheric layers are given for $\Delta \log \theta$ and $B$ and it is concluded that they are small enough for the Boussinesq approximation to hold. However, Schertzer and Lovejoy (1985) pointed out that in fact these quantities are highly intermittent so that “typical” values may not be as representative as expected. Furthermore, they showed that $\Delta \log \theta$ and $B$ also depend systematically on the layer thickness $\Delta z$ so that we must consider $\Delta \log \theta(\Delta z)$, $B(\Delta z) = \Delta \log \theta(\Delta z)/\Delta z$. Using a sample of 80 radiosondes from a scientific campaign in Landes, France, they empirically found the following behaviour for the probability distributions of fluctuations over layers of thickness $\Delta z = \lambda \Delta z_1$;

$$\Pr(s' > s) \approx \left(s \frac{\lambda^H}{\lambda} \right)^{q_0} ; \quad s = \frac{\Delta u(\Delta z_1)}{\Delta u(\Delta z)} ; \quad \lambda = \frac{\Delta z_1}{\Delta z}$$

(3)

where $u$ is an atmospheric field (temperature, horizontal wind etc.), $s$ is the ratio of fluctuations over thick layers $(\Delta z_1)$ to layers $\lambda$ times thinner, and “$\Pr$” indicates “probability”. Note that (3) only holds accurately for the probability tails i.e. $\Pr < 10^{-2}$. This type of behaviour was empirically found over the range 50 m to 3.2 km for $u=\log \theta$, the horizontal wind shear $(u=v)$, the Richardson number $(u=\text{Ri})$, and the pressure. Figure 2(a) shows a graphical representation based on the regressions for $\Delta \log \theta$ published in (Schertzer and Lovejoy 1985).
Figure 2(a). Contour plots of \( \log_{10} Pr \) for \( u = \log \theta \) based on data from Schertzer and Lovejoy: \( H_{\log \theta} = 9/10 \), \( q_{\log \theta} = 10/3 \) and for reference values \( \Delta z_1 = 50m \) and \( \Delta \log \theta (\Delta z_1) = 10^{-3} \). The probability contours 1, 0.1, 0.01… 10\(^{-6} \) are shown (white to black respectively; although the contours are not very accurate for \( Pr > 10^{-2} \)). The vertical line indicates that, for layers 1km thick, fluctuations are almost certain to be greater than 1% and there is roughly a 0.1% chance that fluctuations greater than 10% will occur. The horizontal and vertical coordinates are \( \log_{10} \Delta z \), \( \log_{10}(\Delta \log \theta) \), units are m, dimensionless respectively.

From the parameter values and the figure, we see that fluctuations can be large. However, it is useful to directly calculate the statistics of the static stability \( B(\Delta z) \) for fluctuations over various layers; this is shown in Fig. 2(b). This figure clearly shows the role of intermittency. For example, we find that for 300m thick layers - which is typical for NWP models – 0.1% of the resolution elements will have fluctuations so large that the Boussinesq assumption is violated. Presumably, the more demanding stratification approximations (e.g. hydrostatic) used by typical NWP models will suffer even worse violations. As for the Brunt Väisäla\( \text{"a} \) frequency, it is often assumed that \( N \) is fixed over a given atmospheric layer. Such an approximation might seem reasonable because of the small value of the scaling exponent of \( N \). From Eq. (1), we find \( H_N = (H_{\log \theta} - 1)/2 = -1/20 \), which, although small, is not zero, so that \( N \) does indeed vary with the thickness of the layer. In addition, this small exponent masks large fluctuations which – being highly negatively correlated - tend to cancel each other. While the scaling exponent of \( N \) is
small, the intermittency is not. From Eq. 1, we find \( q_{DN} = 2 q_{DB} \) and \( q_{DB} = 10/3 \), so that within the apparently nearly constant N layer, there are sublayers with large variations.

Figure 2(b). The probability contour plot corresponding to Fig. 2(a) for the static stability; the parameters \( H_B = H_{\log\theta} - 1 \), \( q_{DB} = q_{D\log\theta} = 10/3 \) and for \( \Delta z_1 = 50m \), \( B_1 = 2 \times 10^{-5} m^{-1} \). (corresponding to \( H_N = (H_{\log\theta} - 1)/2 \), \( q_{DN} = 2 q_{DB} \); see (2)). The horizontal and vertical coordinates are \( \log_{10}\Delta z \), \( \log_{10}(\Delta \log\theta/\Delta z) \), units are m, m\(^{-1}\) respectively. The lines demonstrate the example that at a probability 1%, it will breakdown over layers 30cm thick, at probability of 0.1%, it will breakdown over layers 300m thick, probability 0.03%, at layers 3km thick.

Figures 2(a) and (b) examine the static stability but not the dynamical stability. The usual criterion for this is based on the Richardson number (\( Ri \)), which we here define over a layer of thickness \( \Delta z \):

\[
Ri(\Delta z) = g\Delta z \frac{\Delta \log \theta(\Delta z)}{\Delta v(\Delta z)^2}
\]  

(4)

where \( \Delta v \) is the shear in the horizontal velocity and \( g \) is the gravitational constant. Standard theory dictates that a layer is stably stratified only if \( Ri > 1/4 \). In order to examine this empirically, we can again use the analyses in Schertzer and Lovejoy (1985), to plot the probability contours for \( Ri(\Delta z) \) (see Fig. 2(c)).
Figure 2(c). The probability contour plot corresponding to Fig. 2(b) for the Richardson number. Using \( H_{\text{Ri}} = 1, q_{\text{DRi}} = 1 \), we find for \( \Delta z_1 = 50 \), we have \( \Delta \text{Ri}_1 = 0.5 \). The horizontal and vertical coordinates are \( \log_{10} \Delta z \), \( \log_{10} \text{Ri} \) respectively; units are m and dimensionless respectively. This shows that for layers 25 m and thicker, the layer exceeds the Richardson criterion for stability \( \text{Ri} > 1/4 \), whereas the thinner layers have a smaller and smaller chance of being stable. For example, layers 2.5 m thick have an only 10% chance of being stable.

We see from the plot that for layers 25 m and thicker, the Richardson stability criterion is typically exceeded, implying the typical stability of thick layers (actually, the curves for \( \text{Pr} > 10^{-2} \) are not so accurate, but this illustrates the idea). However, overall stability conclusions follow only with additional homogeneity assumptions about the layer which the same scaling law shows is in fact strongly violated. Extending the statistical laws to considering the small scale structures of a thick layer, we find that each layer is composed of sublayers, some of which are almost surely unstable. For example, we find that sublayers 2.5 m thick have an only 10% chance of being stable. From this analysis, we conclude that standard treatments of stratification, while perhaps apparently justified “on average” over “typical” layers, are nevertheless in general unjustified. Since it appears that \( \Delta \log \theta \) occasionally has large fluctuations and \( g \Delta \log \theta \) is the buoyancy force across a layer, we see that buoyancy forces actually play a dominant role in atmospheric dynamics, a fact that is occulted in mainstream approaches which assume dynamical forcings of unspecified origin.

(ii) Anisotropic gravity wave theories. We now turn our attention to anisotropic scaling
Van Zandt (1982) broke with the standard approach in two important respects: a) he was the first to empirically recognize the horizontal/vertical anisotropy of the scaling in the meso-scale and b) based on linear perturbation theory, he postulated that a well-defined linear dispersion relation existed (i.e. a one-to-one relation between the frequency and horizontal and vertical wavenumbers). The consequence of both assumptions was that fluctuations in the horizontal wind could be described by a “universal” anisotropic spectrum. This picture differed from previous isotropic models of (e.g. (Bolgiano 1959; Lumley 1964; Weinstock 1978)) because it did not involve a transition from an isotropic inertial subrange to a buoyancy subrange. In addition, the energy source was not explicit: it was not necessarily turbulent “leakage” as in the Lumley-Shur models. Since VanZandt (1982), almost all empirical observations of the horizontal wind and temperature (and for the upper atmosphere, for the density) have assumed different spectral exponents in the horizontal and vertical directions; they all have $2 < D_s < 3$.

Currently, there are two main approaches used in the literature; the Saturated Cascade Theory (SCT; Dewan & Good (1986), Dewan (1997)) and the Diffusive Filtering Theory (DFT; Gardner (1994)). They both share the key assumptions of Van Zandt (1982) about anisotropic scaling and linear gravity wave dispersion relations. Without going into the theories in detail, let us make a few comments about each. Both theories assume the validity of the classical linear perturbation gravity wave dispersion relation:

$$\omega = \frac{N k_z}{|k|}; \quad |k| = \left( k_x^2 + k_z^2 \right)^{1/2} \ (5)$$

yielding a one-to-one relation between frequency $\omega$ and wavenumber $k$. Equation (5) follows from the Goldman-Taylor equations (in the case where there is no wind). Both the SCT and DFT assume that the waves “saturate” i.e. that instabilities limit the horizontal wind $v_x$ to the horizontal group velocity $v_{gx} = \frac{\partial \omega}{\partial k_x}$ which (for $k_x \ll k_z$) implies the restrictive relation $v_x(k_z) = N k_z^{-1}$ (here and below, we ignore the second horizontal component, $y$). In addition, although the vertical velocity is outside our present scope, both SCT and DFT use the linear theory “polarization” relations to uniquely determine $v_z$ from $v_x$.

At this point the SCT and the DFT differ. In the SCT a (presumably) highly nonlinear wave cascade (not turbulent cascade!) is invoked (Dewan 1997). Since this is controlled by $\varepsilon$, dimensional analysis yields the classical $v(\omega) = \varepsilon^{1/2} \omega^{-3/2}$ where $v$ is the amplitude of the wave at frequency $\omega$. As a result the model predicts the following energy spectra for the horizontal wind:

$$E(\omega) = \varepsilon \omega^{-2}; \quad E(k_z) = \varepsilon^{2/3} k_z^{-5/3}; \quad E(k_z) = N^2 k_z^{-3} \ (6)$$

(we will not discuss the predictions for the vertical wind or other quantities). The first two spectra are in fact the classical Kolmogorov relations (although for anisotropic turbulence) whereas the key third prediction $E(k_z) = N^2 k_z^{-3}$ (as noted in (Dewan 1997))
actually follows directly from dimensional analysis if \( N \) is assumed to be the unique parameter controlling the vertical structure. This fact suggests that many other (less restrictive) derivations are possible.

In comparison, the general DFA postulates a key dynamical and spectral role for three scale parameters: the inertial frequency \( f (\approx 10^{-4} \text{s}^{-1}) \), the Brunt- Väisälä frequency \( N < f \) and a vertical wavenumber \( k_z^* \) (estimated empirically as \((15 \text{ km})^{-1}\) in Hostetler & Gardner (1994)). Together, these can be used to nondimensionalize the spectra to obtain:

\[
E(\omega) \propto \left(\frac{\omega}{f}\right)^{-p} \quad \omega < f
\]

\[
E(k_z) = \left(\frac{k_z}{k_z^*}\right)^{-q} \quad k_z < k_z^*
\]

where \( p \) and \( q \) are arbitrary. From here, the saturation and dispersion conditions are used to obtain \( E(k_z) \) which has a complicated form with three regimes including notably a break at \( k_z = k_z^* \). Note that this argument implies that the low frequency, wavenumber parameters \( f, k_z^* \) determine the dynamics at the higher ones (at the smaller scales). This is contrary to the usual turbulence phenomenology where the outer scale of the turbulence only determines the intermittency properties (e.g. the corrections to the basic Kolmogorov law), but not the form of the law itself (i.e. not the 5/3 value). Although \( p \) and \( q \) can be arbitrary, (Gardner et al. 1993) use a wave saturation argument combined with an assumption about the bandwidth to argue that \( q = 3 \) as in the SCT. They also argue that empirically \( p \approx 2 \) so that in practice - as far as the empirically accessible exponents of the horizontal wind is concerned - the SCT and DFA differ primarily in their predictions for \( E(k_z) \) (the former is scaling with \( \beta = 5/3 \), the latter has 3 different regimes).

The basic difficulty with both theories is that in order for linear gravity waves to be meaningful, the nonlinearity must be weak enough so that linear dispersion relations can be defined, while if the spectra have turbulent exponents (and are presumably turbulence driven), then the nonlinearity must be strong. In the next section on the 23/9 model we shall see that if one uses the turbulent flux (\( \phi \)) in place of \( N \) (leading to \( E(k_z) = \phi^{5/5} k_z^{-11/5} \)), then one can obtain a “turbulent” dispersion relation and the observables such as \( v, \rho \) can be obtained directly from turbulent sources with wave-like propagators. The theoretical advantage is that all the standard estimates of nonlinearity (such as Reynolds number) show it to be extremely high and \( \phi \) is indeed a quadratic invariant as required for cascading turbulent fluxes.

3. THE 23/9D FRACTIONALLY INTEGRATED FLUX MODEL

(a) The 23/9 D model

(i) The basic assumptions. As was mentioned already, the fundamental driving mechanism of atmospheric dynamics is buoyancy, and buoyancy introduces a second quadratic invariant (the flux \( \phi \)), in addition to the energy flux \( \varepsilon \). While the isotropic
Bolgiano-Obukhov theory does place buoyancy in a central role, for scales larger than the small Bolgiano length $l_B$ it unfortunately relegates the energy flux to a secondary role, whereas at least in the horizontal the data clearly show $k_x^{-5/3}$ out to large scales (part II).

By the early 1980’s it had become clear that - at least in the horizontal direction and in time - the horizontal wind spectrum is close to the Kolmogorov $k^{-5/3}$ form out to at least tens if not hundreds of kilometers. Neither isotropic $k^{-11/5}$ nor isotropic $k^{-3}$ spectra were compatible with this, precluding the existence of an isotropic Bolgiano-Obukhov buoyancy subrange or that of an isotropic gravity wave dominated atmosphere as first suggested by Lumley-Shur (1963) and Weinstock (1978). In addition, the problems inherent in in-situ or aircraft measurements were avoided by the analysis of perimeters of satellite cloud and radar rain areas spanning the scales of 1 to over 1000 km (Lovejoy 1982) that indicated the existence of single scaling regime in the horizontal right through the mesoscale (presumably corresponding to a $k^{-5/3}$ spectrum in the horizontal).

These satellite analyses combined with earlier $k^{-11/5}$ spectra obtained by radiosonde observations of horizontal wind shear along the vertical made by Endlich et al. (1969) and Jimsphere observations by Adelfang (1971) provided the context of the Landes experiment discussed in Section (2.b.1) and lead to the anisotropic 23/9 “unified scaling” model (Schertzer & Lovejoy 1983a; Schertzer & Lovejoy 1985a). Since the theory was designed to explain the observed differential stratification over a wide range of scales, it was important that it not require the Boussinesq (and similar) approximations. Rather, it was directly based on the buoyancy force variance flux $\phi = \Delta f^2 / \tau$ (units of distance$^2$/time$^5$) in the vertical, where $\Delta f = g \Delta \log \theta$, is the buoyancy force gradient across a layer thickness $\Delta z$, and $\tau$ is the time scale of the transfer. Although $\Delta f$ and $\phi$ have the same units as in the Bolgiano-Obukhov theory, here $\Delta f/\Delta z$ is the actual buoyancy force felt by a parcel of air along the vertical and not a theoretical force defined in reference to a mean level (as in the Boussinesq and related approximations). At the same time the model supposes that the horizontal structure is dominated by an energy flux $\varepsilon = \Delta v^2 / \tau$ where $\Delta v / \Delta x$ is a horizontal shear in the horizontal wind, and the time scale $\tau = \Delta x / \Delta v$.

From dimensional analysis on the basic fluxes $\phi$ and $\varepsilon$, one obtains unique length and time scales:

$$l_B = l_s = \phi^{-3/4} \varepsilon^{5/4}, \quad \tau_B = \tau_s = \phi^{-1/2} \varepsilon^{1/2} \quad (8)$$

The subscript “$B$” is used for “Bolgiano” because analogous scales are introduced in the Bolgiano-Obukhov and other buoyancy subrange theories (where $\phi$ is defined instead with the help of the Boussinesq approximation). However, the important difference is that in the latter theories, they denote transition scales between two isotropic regimes (a small scale isotropic $\varepsilon$-dominated regime and a large scale isotropic $\phi$-dominated regime). In contrast, in the 23/9 D model, there is no qualitative change in behaviour; $\phi^{-3/4} \varepsilon^{5/4}$ is simply the scale at which structures are roughly isotropic, it is the “sphero-scale” and hence we prefer the notation $l_s$ with $\tau_s$ as the corresponding “sphero-time”; the lifetime of structures in the flux field whose size is the sphero-scale. While $l_B, \tau_B$ of the original buoyancy subrange theories were never observed, the only direct measurements of $l_s$ are aircraft based ($l_s = 4$ cm for wind in the stratosphere, (Lovejoy et al. 2004)) and for lidar aerosol data $l_s \approx 4 - 80$ cm (with mean around 10cm, see (Lilley et al. 2004); see
also (Tuck et al. (2004)). Both results are close to the values of \( l_b \) theoretically estimated to be of the order of 1 m. We shall see that the turbulent flux \( \phi \) replaces \( N \) as the basic dimensional parameter in the vertical so that the sphero-frequency \( \omega_s = 1 / \tau_s = \phi^{1/2} \varepsilon^{-1/2} \) replaces \( N \) as the basic (flux dependent) time scale. Although there have been no direct estimates of \( \omega_s \), we can interpret the gravity wave based estimate of the buoyancy frequency in (Allen & Vincent 1995) of roughly \((40 \text{ s})^{-1}\) as an order of magnitude for \( \omega_s \). The corresponding sphero-velocity is thus of the order of magnitude \( l_s \omega_s \approx 2 \text{ mm s}^{-1} \) although it is highly variable since it depends on the turbulent fluxes \( \varepsilon, \theta \).

(ii) **Anisotropic scaling and scale functions.** The basic hypothesis is that \( \varepsilon \) dominates the horizontal and \( \phi \) the vertical so that horizontal wind shears follow:

\[
\begin{align*}
\Delta v(\Delta x) &= \varepsilon^{1/3} \Delta x^{1/3} \\
\Delta v(\Delta y) &= \varepsilon^{1/3} \Delta y^{1/3} \\
\Delta v(\Delta z) &= \phi^{1/5} \Delta z^{3/5} \\
\Delta v(\Delta t) &= \varepsilon^{1/2} \Delta t^{1/2}
\end{align*}
\]

(9)

The first two describe the real space horizontal Kolomogorov scaling, the third the vertical Bolgiano-Obukhov scaling for the velocity. The anisotropic Corrsin-Obukhov law (see parts II, III) is obtained by the replacements \( v \to \rho, \quad \varepsilon \to \chi^{3/2} \varepsilon^{-1/2} \) where \( \chi \) is the passive scalar variance flux. We have included the fourth line which is the classical result for the pure time evolution in the absence of an overall advection velocity; this is the classical Lagrangian version of the Kolmogorov law. In parts II, III, we indeed show that the anisotropic Corrsin-Obukhov version of (9) holds very accurately. Finally, below we show how to modify the above to take into account advection.

The question about Kolmogorov versus Bolgiano-Obukhov scaling is also debated in laboratory buoyancy driven turbulence flows (e.g. Benard convection). The 23/9D theory may apply there as well as in the atmosphere, although at the moment the crucial vertical spectrum of the horizontal velocity has not been adequately investigated (see the discussion in (Lilley et al. 2004)).

Since there is no characteristic scale, we expect generalized scale changes to define mathematical groups with generators \( G \). Using this idea, the scaling (9) can be combined into a single expression valid for any space-time vector displacement \( \Delta \mathbf{R} = (\Delta x, \Delta y, \Delta z, \Delta t) \) by introducing a space-time scale function \( \| \Delta \mathbf{R} \| \), which satisfies the fundamental scale equation (Schertzer and Lovejoy 1985, 1987):

\[
\left[ T_\lambda \| \Delta \mathbf{R} \| \right] = \lambda^{-1} \left[ \| \Delta \mathbf{R} \| \right]
\]

(10)

where the scale changing operator \( T_\lambda \) is a continuous one-parameter (Lie) group with generator \( G \) such that:

\[
T_\lambda = \lambda^{-G}
\]

(11)

Note the notation “\( \| \)" for the space-time scale function is distinguished from “\( \| \)" which is the spatial part (see below). This is the basic framework for defining scale in an
anisotropic scaling system. When \(G\) is a matrix, the notion of scale is position independent: for the case in which \(G\) is the identity matrix, we have the usual isotropic, self-similar scale changes. In the case of "linear GSI", where \(G\) is a diagonal matrix, the system is "self affine" and we obtain stratification along a coordinate axis. Finally, when \(G\) has off-diagonal elements we have differential rotation and stratification.

To unify horizontal, vertical and temporal turbulent fluctuations as described in (9), we require:

\[
G = \begin{pmatrix}
G_x & 0 & 0 \\
0 & G_y & 0 \\
0 & 0 & G_z
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where the rows and columns correspond to the \(x, y, z\) and \(t\) directions respectively, \(G_i\) is the matrix corresponding to the spatial part only; we define \(D_s = \text{Trace } G\) and \(D = \text{Trace } G = D_s + H_t\) as the elliptical dimensions characterizing the spatial and space-time anisotropies respectively. With the above dimensionally determined exponents we find \(D_s = 23/9\), \(D_s = 29/9\) (space, space-time respectively). For an arbitrary space-time displacement \(\Delta \mathbf{R} = (\Delta r, \Delta t)\), we may now write

\[
\Delta v(\Delta \mathbf{R}) = \mathcal{E}_{[\Delta \mathbf{R}]}^{1/3} \|\Delta \mathbf{R}\|^{1/3}
\]

where the subscripts on the flux indicate the space-time scale over which it is averaged.

Let us now consider only the spatial scale function \(\|\Delta \mathbf{r}\|\) which satisfies the equation:

\[
\hat{\lambda}^{-G} \Delta \mathbf{r} = \hat{\lambda}^{-1} \|\Delta \mathbf{r}\|
\]

A particularly simple (but by no means unique) scale function which we call the "canonical scale function" is introduced by the following nonlinear transformation of variables:

\[
x' = x; \\
y' = y; \\
z' = l_s \left| \frac{z}{l_s} \right|^{H_t} \text{ sign}(z)
\]

\[
\|\Delta \mathbf{r}\|_{c an} = (\Delta \mathbf{r}' \cdot \Delta \mathbf{r}')^{1/2} = l_s \left( \left( \frac{\Delta x}{l_s} \right)^2 + \left( \frac{\Delta y}{l_s} \right)^2 + \left( \frac{\Delta z}{l_s} \right)^{2/H_t} \right)^{1/2}
\]

It is easy to check that, by setting \(\Delta \mathbf{r} = (\Delta x, 0, 0), \Delta \mathbf{r} = (0, \Delta y, 0), \Delta \mathbf{r} = (0, 0, \Delta z), (13), (15)\) reduce to the first three equations (9). In the nonlinearly transformed \(\Delta \mathbf{r}'\) space, the vector \(\Delta \mathbf{r}'\) satisfies the scale equation (14) but with \(G_s' = \text{identity}\). Therefore we see that one family of solutions of (14) is:
\[ \|\Delta \mathbf{r}\| = \Theta(\Omega') \|\Delta \mathbf{r}\| \]  
\[ (16) \]

where \( \Omega' \) is a unit vector of spherical polar angles in the three dimensional \((x', y', z')\) space. The family of scale functions defined in this way are “physical scale functions” since in addition to (14), they satisfy a basic “localization” requirement of physical scale:

\[ B_\lambda \subseteq B_\lambda' \iff \lambda \geq \lambda'; \ \{ B_\lambda; \|\mathbf{r}\| \leq \lambda \} \]
\[ (17) \]

where \( B_\lambda \) is the set of all the vectors with scale not exceeding \( \lambda \). Scale functions satisfying condition (17) define “balls” (basic sets) \( B_\lambda \) which are strictly decreasing functions of scale and therefore can be used to define anisotropic Hausdorff measures and hence the notion of spatial integration (Schertzer & Lovejoy 1985b), and anisotropic Hausdorff dimension.

We can now use our physical spatial scale function \( \|\Delta \mathbf{r}\| \) to define a localized space-time scale function \( \lambda^{G} \Delta \mathbf{R} \) which satisfies \( \lambda^{G} \Delta \mathbf{R} = \lambda^{-1} \lambda^{G} \Delta \mathbf{R} \):

\[ \|\lambda^{G} \Delta \mathbf{R}\| = l_s \left( \left\| \lambda^{G} \Delta \mathbf{R}\right\| \right)^{2} + \left( \left| \lambda^{G} \Delta \mathbf{R}\right| \right)^{2/\gamma} \frac{1}{\tau_s} \]  
\[ (18) \]

(see Marsan et al 1996). While this special form will be retained for the turbulent fluxes: we are not so strongly restricted; we shall see that more general space-time scale functions can be used leading to (unlocalized) wave behaviour.

Finally, before proceeding, we recall that we have ignored advection. As pointed out in (Schertzer et al. 1997b), advection can be taken into account using the Gallilean transformation matrix \( A \):

\[ A = \begin{pmatrix} 1 & 0 & 0 & -v_x \\ 0 & 1 & 0 & -v_y \\ 0 & 0 & 1 & -v_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  
\[ (19) \]

The new generator \( G' = A^{-1} G A \) and the corresponding scale function \( \lambda^{G} \Delta \mathbf{R}_{\text{adv}} \) which is symmetric with respect to \( G' \) is \( \lambda^{G} \Delta \mathbf{R}_{\text{adv}} = \lambda^{-1} \lambda^{G} \Delta \mathbf{R} \).

\( (b) \) Intermittency and the Fractionally Integrated Flux model
(i) Introduction. Starting with (Novikov & Stewart 1964; Yaglom 1966; Mandelbrot 1974), stochastic isotropic cascade models have been developed for modeling intermittent $\epsilon$ fields. In the 1980’s it was realized that these cascades were multifractal i.e. the increasingly intense turbulent regions are concentrated on sparse fractal sets with dimension decreasing with increasing $\epsilon$. Since the nonlinear terms in the Navier-Stokes equations conserve $\epsilon$, it is modeled by a “conservative” cascade i.e. the mean is independent of scale.

Our object is to simulate a component of the horizontal velocity field satisfying (13) and respecting causality. The first step is to give (13) a statistical interpretation; it is usual to assume that it only states an equality in the scaling of $\Delta v(\Delta r)$ and $\epsilon^{1/3} [\Delta R]^{1/3}$

so that taking the ensemble average of $q$th powers (13) we have:

$$\langle |\Delta v|^q \rangle = \langle \epsilon^{q(1/3)} [\Delta R]^{q(1/3)} \rangle ; \quad \epsilon(q) = qH - K(q/3) ; \quad H = 1/3$$

where $K(q)$ is the nonlinear exponent characterizing the intermittency of $\epsilon$. Due to the existence of attractive, stable multifractal processes (Schertzer & Lovejoy 1997); $K(q)$ is expected to take a special “universal” form characterized by two parameters

$$K(q) = \frac{C1}{\alpha - 1}(q^\alpha - q)$$

where $0 < C1 < D$ is the codimension of the mean field, and $0 \leq \alpha \leq 2$ is the index of the Levy noise generator (see below); empirically, we find $C1 \approx 0.25$, $\alpha \approx 1.5$ for $\epsilon$ (see e.g. Schmitt et al. 1996).

(ii) The fractionally integrated flux (FIF) model. We now describe the Fractionally Integrated Flux (FIF) model (Schertzer & Lovejoy 1987) which satisfies the anisotropic Kolmogorov law (13) and the multiscaling statistics (20), (21). One starts with a subgenerator $\gamma_\alpha(r,t)$ (whose normalization depends on $C1$ and form of the probability distribution depends on $\alpha$) which is a noise composed of independent identically distributed Levy random variables (the special case $\alpha=2$ is the Gaussian leading to the “log-normal” multifractals). One next produces the generator $\Gamma(r,t)$ by convolving (“*”) this with the scaling propagator (the space-time Green’s function to which we return momentarily) $g_\alpha(r,t)$:

$$\Gamma(r,t) = \gamma_\alpha(r,t) * g_\alpha(r,t) ; \quad \Gamma(k,\omega) = \gamma_\alpha(k,\omega) \tilde{g}_\alpha(k,\omega)$$

where we have indicated fourier transforms by tildas. The conserved flux $\epsilon$ is then obtained by exponentiation:

$$\epsilon(r,t) = e^{\Gamma(r,t)}$$

The horizontal velocity field is obtained by a final convolution with the (generally different) propagator $g_r(r,t)$:

$$v(r,t) = \epsilon^{1/3}(r,t) * g_r(r,t) ; \quad \nu(k,\omega) = \epsilon^{1/3} \tilde{g}_\alpha(k,\omega)$$

$$v(r,t) = e^{\nu(r,t)}$$

$$v(r,t) = \epsilon^{1/3}(r,t) * g_r(r,t) ; \quad \nu(k,\omega) = \epsilon^{1/3} \tilde{g}_\alpha(k,\omega)$$

$$v(r,t) = e^{\nu(r,t)}$$
In order to satisfy the scaling symmetries, it suffices for the propagators to satisfy the generalized scale equation:

$$g (T_\lambda (\Delta \mathbf{r}, \Delta t)) = \lambda^{D-H} g (\Delta \mathbf{r}, \Delta t)$$

where $H$ must be chosen $= D(1-1/\alpha)$ for $g_x$ and $H=1/3$ for $g_v$ (recall $D=\text{Trace}(G)$ is the “elliptical dimension characterizing the overall stratification of space-time). The solutions of (25) are powers of scale functions with a Heaviside function $h(t)$ needed to ensure that causality is respected:

$$g (\Delta \mathbf{r}, \Delta t) = h(t) \lambda^{D-H}; \quad h(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

(c) An extreme unlocalized (wave) model

(i) Waves/turbulence model. Although the FIF is quite general, its classical implementation (see e.g. (Schertzer et al. 1997a), (Marsan et al 1996)) is obtained by using the same localized space-time scale function for both $g_x$ and $g_v$. Such propagators are “power law” localized in space-time (recall that the term “localization” is often taken to mean a much stronger exponential localization). For $g_x$ this localization is justified by the usual turbulence phenomenology – that turbulent energy fluxes are indeed localized (as required for example by the Navier-Stokes equations). However, we shall now consider the possibility that the propagator $g_v$ is nonlocal in space-time (although still with some localization in space; wave packets). The key is to recognize that the physical spatial scale function $||\mathbf{r}||$ defines another dual fourier scale function $||\mathbf{k}||$ satisfying (13) but with generator $G_{sT}$ (“T” indicates the transpose). Indeed, more precisely:

$$||\mathbf{k}||^{-H}_{\text{F.T.}} \leftrightarrow ||\mathbf{r}||^{-(D-H)}; \quad D_s = \text{Trace} G_s$$

where “F.T.” indicates “Fourier Transform”. Note that in terms of space-time scale functions we also have:

$$[\mathbf{k}, \omega]^{-H}_{\text{F.T.}} \leftrightarrow h(t)[\mathbf{r}, t]^{-(D_s-H)}; \quad ||\mathbf{k}|| = [\mathbf{k}, 0]; \quad D_{st} = D_s + H_t$$

Equations (27), (28) indicate that the fourier transforms of generalized power laws with respect to $G_{sT}$ are in turn generalized power laws with respect to $G_{st}$, i.e.

$$[\lambda^{-G_{st}} (\mathbf{r}, t)] = \lambda^{-1} [\mathbf{r}, t] \quad \text{and} \quad [\lambda^{-G_{st}} (\mathbf{k}, \omega)] = \lambda^{-1} [\mathbf{k}, \omega].$$

This is an anisotropic extension of classical “Tauberian theorems” (see e.g. (Feller 1971)). Note that in spite of the notation, the fourier and real space scale functions are generally different.

Due to (25), (28), the velocity propagator must be chosen in fourier space to respect the appropriate scaling symmetries:

$$g_v (\mathbf{k}, \omega) = [\mathbf{k}, \omega]^H$$

However, the key to the extreme unlocalized (wave) model is to choose
\[
[k, \omega] = \left(i(\omega - \|k\|^{H_t})\right)^{1/H_t}
\] (30)

In order to understand the implications of this scale function, it is instructive to take the inverse Fourier transform with respect to \(\omega\):
\[
\tilde{g}_v (k, t) = h(t) t^{-1+H_t/H_c} \, e^{H_t t}
\] (31)

(we ignore constant factors). This shows that the propagator defined by (29), (30) is a causal (due to \(h(t)\)) temporal fractional integration of order \(H_t/H_c\) of waves. Indeed, we can use the standard method of stationary phase (e.g. (Bleistein & Handelsman 1986)) to obtain an asymptotic approximation to the space-time convolution for \(v\) (24):
\[
v(x, t) \approx e^{1/3} (k(x), t)^* \, \tilde{g}_v (k(x), t)
\] (32)

where “\(\ast\)”, indicates convolution with respect to time only and the propagator is:
\[
g_v (x, t) \approx \tilde{g}_v (k(x), t) = h(t) e^{i[k \cdot x - \omega_d(k)t + \phi(k)]}
\frac{i^{1/2 - H_t/H_c} \det \left[ \frac{\partial^2 \omega_d(k)}{\partial k_i \partial k_j} \right]}{t^{1/2 - H_t/H_c}}
\] (33)

where \(\phi\) is a phase and:
\[
x = v_g (k) t; \quad v_g (k) = \nabla \omega_d (k); \quad \omega_d (k) = \|k\|^{H_t}
\] (34)

Equation (33) should be understood as a parametric equation: \(k\) is the wavevector which satisfies the “ray” equation \(x = v_g (k) t\), where \(v_g\) is the group velocity and \(\omega_d (k) = \|k\|^{H_t}\) is the dispersion relation. The above shows that the velocity field is the fractional time integral of wave packets propagating along rays at the group velocity, dispersing and decreasing in amplitude as they travel. The classical time dependence of the attenuation of the packet is \(t^{-3/2}\) so that the waves attenuate a little faster (with \(H=1/3\), \(H_t=2/3\), the exponent is \(5/2 - H = 2 > 3/2\)). Also, as usual, the above breaks down when the determinant in the denominator vanishes; these singular curves are the “caustics”. Table 1 shows the comparison of the turbulent and velocity propagators.
Flux (Power law!) Localization in space-time

| e.g. $||k, \omega|| \approx \hbar * (|\omega| + ||k||^2)^{1/H}$ |
|---|
| $g_{e}(k, \omega) = ||k, \omega||^H$ |

Wave-like velocity Unlocalized

<table>
<thead>
<tr>
<th>e.g. $f(z) \rightarrow z^{H/H-1}$; $f(z) \rightarrow 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{w}(k, t) = h(t) t^{1+H/H}$, $f(z)$</td>
</tr>
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</table>

Table 1. An intercomparison of flux and wavelike velocity propagators. In both cases, the physical (spatial) scale function $||r||$ and fourier counterpart $||k||$ are the dual and are linked by (27).

(ii) Gravito-turbulence dispersion relations. The standard gravity wave model assumes for a layer thickness $\Delta z$, a uniform stratification characterized by

$$N^2 = g \left( \frac{\Delta \log \theta}{\Delta z} \right),$$

and a separation of scales allowing a linear perturbation analysis. The resulting Taylor-Goldman equations then lead to the dispersion relation (5) (we discuss only its simplest form which is quite adequate for our purpose). In contrast, our turbulence flux based approach assumes a highly heterogeneous vertical structure whose statistics are controlled by the buoyancy variance flux

$$\phi = g \left[ \frac{(\Delta \log \theta)^2}{\tau_b} \right] \equiv_{0,\Delta z}$$

(the subscript indicates that the flux is measured at resolution $\|0,0,\Delta z\|$; when needed, the “turbulent” buoyancy frequency is $\omega_t = (\phi/\epsilon)^{1/2}$). The combined $\epsilon, \phi$ fluxes lead to a physical scale function $||r||$, and thus to the dispersion relation:

$$\omega = ||k||^H; \quad ||k||^H \leftrightarrow \omega^{(D-H)}$$

(35)
However, the scale function is fairly general. For example, considering only the vertical $(x,z)$ plane, it is of the form:

$$\|k\| = \tilde{\Theta}(\Omega')\|k\|_{\text{can}}; \quad \|k\|_{\text{can}} = I_{x^0}^{-1}\left(\left(k_x I_x \right)^2 + |k_z I_z|^{2/H_z}\right)^{1/2}$$

(36)

where $\tilde{\Theta}(\Omega')$ is a relatively arbitrary function of direction in the vertical plane $(k_x', k_z')$; $\Omega' = \arctan(k_x'/k_z')$; see (15), (16). Note that in general the fourier scale function is symmetric with respect to $G_x^T$, not $G_z$ (this is important only when $G_z$ has off-diagonal elements – as for example when there is an overall advection velocity, we ignore this in (33) which is thus symmetric with respect to $T_x = \lambda^{-G_z}$; $G_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & H_z \end{pmatrix}$).

Several of the predictions of gravity wave theory have been at least roughly empirically verified; it is therefore of interest to choose $\tilde{\Theta}(\Omega')$ so that the turbulence/wave theory gives a similar dispersion relation and hence gives similar predictions. Since the classical dispersion relation is symmetric with respect to isotropic scale changes (i.e. with $x$, $z$ generator $G_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ rather than the anisotropic $G_z = \begin{pmatrix} 1 & 0 \\ 0 & H_z \end{pmatrix}$), the two dispersion relations cannot be identical. However, they can be chosen to be sufficiently similar so that the new relation can plausibly be compatible with the results of previous atmospheric gravity wave studies.

Using the values $H_z=2/3$, $H_z=5/9$ from the dimensional analysis, we find that the choice $\tilde{\Theta}(\Omega') = (\cos\Omega')^{1/2}$ leads to the following gravity wave-like “gravito-turbulent” dispersion relation:

$$\omega(k) = \varepsilon^{1/3} \frac{|k_z|}{\|k\|^{1/3}}$$

(37)

To display the similarity with the classical dispersion relation more clearly, table 2 shows the two special cases which are most commonly tested empirically: near horizontal and near vertical propagation.
Linear theory gravity wave dispersion | “Gravito-turbulent” dispersion
---|---
**General form** |
\[
\omega(k) \approx g^{1/2} \left( \frac{\Delta \log \theta}{\Delta z} \right)^{1/2} \frac{|k_z|}{|k|};
\]
\[
|k| = \left( k_x^2 + k_z^2 \right)^{1/2}
\]

\[
\omega(k) = \omega^ {1/3} \left( \frac{|k_z|}{|k|} \right)^{1/3};
\]
\[
|k| = l_s^{-1} \left( (l_s k_x)^2 + (l_s k_z)^{18/5} \right)^{1/2};
\]

\[
l_s = \phi^{3/4} \varepsilon^{5/4}
\]

**Near horizontal propagation** |
\[
\omega(k) \approx g^{1/2} \left( \frac{\Delta \log \theta}{\Delta z} \right)^{1/2} \frac{|k_x|}{|k|};
\]
\[
|k_x| << |k_z|
\]

\[
\omega(k) = g^{2/5} \left( \frac{\Delta \log \theta}{\Delta z} \right)^{2/5} \frac{|k_x|}{|k|}^ {2/5};
\]
\[
|k_x| << |k_z|^{9/5} l_s^{4/5}
\]

**Near vertical propagation** |
\[
\omega(k) \approx g^{1/2} \left( \frac{\Delta \log \theta}{\Delta z} \right)^{1/2} ;
\]
\[
|k| >> |k_z|
\]

\[
\omega(k) = \omega^ {1/3} \left( \frac{|k_z|}{|k|} \right)^{2/3};
\]
\[
|k_z| >> |k|^{9/5} l_s^{1/5}
\]

| Table 2. A comparison of the standard gravity wave dispersion relations with a turbulent/wave model with a gravity wave like choice of \( \tilde{\Theta}(\Omega') \); a "gravito-turbulence" dispersion relation. To make the comparison more clear, we have expressed the flux \( \phi \) in terms of the potential temperature and \( g \). Recall that \( l_s \) is the sphero-scale; it’s value is of the order of 10 cm. |

It can be seen that in both cases, for horizontal propagation, the dispersion relation becomes linear in \( k_x \) so that the horizontal group velocity is independent of \( k_x \), i.e. it “saturates”. In addition, the dependencies on \( \Delta \log \theta \) are very similar (a 2/5 power instead of a ½ power) although it should be recalled that in the turbulence case, the potential temperature profile is considered highly variable, not linear. Also, for near vertical propagation, in both cases \( \omega \) is independent of \( k_z \). A final physically significant similitude is the fact that in both cases, the group velocity has a “restoring” vertical component i.e. \( v_{gz} \) is opposite in sign to \( k_z \) so that packets of waves with upward pointing wavevectors move downwards and packets with downward pointing wavevectors move upwards (all this in the absence of wind). The comparison of the group velocities is shown in Fig. 3. In Fig. 4 we show \((x,z)\) and \((t,z)\) sections of \((x,z,t)\) numerical multifractal simulations showing the stratified wave-like structures that the model produces, including in the effect of advection. In Fig. 5 we show a time sequence, and in Fig. 6 we show the effect of changing the sphero-scale and shape of the unit ball (all these use the grativo-turbulence dispersion/scale function). Finally in Fig. 7 we show simulations of horizontal sections with varying scale functions/dispersion relations, showing that quite realistic morphologies are readily produced.
Figure 3. This shows the contour lines of $\omega$ and arrows the corresponding gradient (group velocities). The formula for the gravito-turbulence dispersion relation waves is the same as in the text (table 2), with $N_s = \omega_s = (\phi/\varepsilon)^{1/2}$. Note that $N_s l_s^{2/3} = \phi^{1/5}$. 

\[
\omega(k) = N \left| \frac{k_x}{|k|} \right|
\]

Gravito-waves

\[
\omega(k) = N_s l_s^{2/3} \left| \frac{k_x}{|k|^{1/2}} \right|
\]

Gravito-Turbulence waves
Figure 4. This figure shows a multifractal simulation of a passive scalar in (x,z,t) space with the observed multifractal parameters ($\alpha=1.8$, $C=0.05$, see parts II, III) and theoretical values $H_t=2/3$, $H_z=5/9$. The simulations show the vertical wind increasing from 0 left to 0.25 to 0.5 pixels/ time step (only a single time step is shown). The top row shows the dispersion relation, group velocity, the second row is an (x,z) cross-section while the third row is a (t,z) cross-section. The simulation method is a development of that described in (Schertzer & Lovejoy 1987; Wilson et al. 1991; Pecknold et al. 1993; Marsan et al. 1996) with numerical refinements to be published.
Figure 5. This shows 8 time steps in the evolution of the vertical cross section of a passive scalar component from the zero wind case of Fig. 4. The structures are increasingly stratified at larger and larger scales, displaying wave phenomenology.
Figure 6. This figure shows the effect of vertical wind (left to right 0, 0.25, 0.5 pixels/time step), and spherico-scale increasing top to bottom from 1 pixel to 4, 16, 64 pixels for vertical cross-sections of simulated passive scalar. For the gravito-turbulence dispersion relation described in the text. The rendering was made using simulated single scattering visible radiation.
Figure 7. This is a series of horizontal sections of \((x,y,t)\) passive scalar cloud simulations with horizontal generator 
\[
G = \begin{pmatrix} 1.2 & 0.05 \\ -0.05 & 0.8 \end{pmatrix}
\]
with \(H_t = 2/3\) as usual. From left to right, the horizontal spheroid-scale = 1 pixel, 8, 64 pixels. The horizontal unit ball is characterized by 
\[
\Theta(\Omega) = 1 + a \cos(2\Omega - 2\Omega_0)
\]
with \(a=0.65\), and from top to bottom, the orientation \(\Omega_0\) is varied from 0 to \(5\pi/6\) (this is the real \((x,y)\) space function c.f. (16)). These simulations show how sensitive the morphologies are to the unit balls (i.e. the spatial scale function/dispersion relation).
(iii) Statistical properties. The basic real space statistical properties of the model follow as in the classical FIF (see Schertzer et al. 1997a); the fields obey (20). The 1D spectra then followed by fourier transforming (20) for $q=2$ taking $\Delta R=(\Delta x,0,0,0)$, $\Delta R=(0,\Delta y,0,0,0)$, $\Delta R=(0,0,\Delta z,0)$, $\Delta R=(0,0,0,\Delta t)$ for $E(k_x)$, $E(k_y)$, $E(k_z)$, $E(\omega)$ respectively. It is however instructive to derive the result by starting with the joint spectrum $P_{v}(k,\omega) = \left\langle |\tilde{v}(k,\omega)|^{2} \right\rangle$:

$$
P_{v}(k,\omega) = \frac{P_{e}(k,\omega)}{(\omega - \|k\|^{2})^{2H/H_{t}}}; \quad P_{e}(k,\omega) = \left\langle |\tilde{e}(k,\omega)|^{2} \right\rangle = \left[\|k\|^{2}\right]^{s}; \quad s = D_{st} - K(2/3) \quad (38)
$$

where $\left[\|k\|^{2}\right]^{s}$ is the fourier counterpart of the turbulent space-time scale function, “symmetrized” by the modulus operation involved in calculating the spectral energy density (see the appendix part III). $D_{st} = \text{Trace}(G_{st}) = 2 + H_{t} + H_{t}$ is the “elliptical dimension” of space-time and $K_{d}(2/3)$ is the intermittency correction for the flux estimated empirically to be $\approx 0.07$ (note the sign, see e.g. Schmitt et al. (1992)); $K_{d}(2/3)$ is proportional to the second moment of $v$ which is proportional to $e^{2/3})$. To determine the spectra with respect to $k_{x}$, $k_{y}$, $k_{z}$, $\omega$, we successively integrate $P_{v}$ with respect to the remaining variables. The results are the classical Kolmogorov and Bolgiano-Obukhov statistics with multifractal (intermittency) corrections:

$$
E(k_{x}) \propto \varepsilon^{2/3} k_{x}^{-5/3} \left( \frac{k_{x}}{k_{1}} \right)^{-K_{d}(2/3)}
$$

$$
E(\omega) \propto \varepsilon^{2/3} \frac{\omega}{\omega_{t}} \left( \frac{\omega}{\omega_{t}} \right)^{-K_{d}(2/3)/H_{t}}
$$

$$
E(k_{z}) \propto \phi^{2/5} k_{z}^{-11/5} \left( \frac{k_{z}}{k_{1}} \right)^{-K_{d}(2/3)/H_{t}} \quad (39)
$$

We have ignored dimensionless numerical constants. The derivation of the spectra in this way has the advantage that the condition for the convergence is clear: we must have $2H - H_{t}$ (due to the singular propagator (30)). When $2H > H_{t}$, Parseval’s theorem proves that the real space variance also diverges, hence we expect a divergence of second order moments for $v$. This is indeed confirmed numerically. Interestingly, the borderline case $2H = H_{t}$ is the one apparently relevant (since $H=1/3$, $H_{t}=2/3$ at least for the extreme wave model presented here, see section 3.b.4) so that the question of divergence/convergence depends on more detailed properties of the propagator than those considered here (there is empirical evidence that the 5th and 7th moments of $v$ diverge in the vertical and horizontal directions respectively (Schertzer & Lovejoy 1985a; Schmitt et al. 1994)). Note that even if the ensemble averaged spectrum does indeed diverge, empirical spectra will converge; they will however display large realization to realization fluctuations – perhaps not unlike real radiosonde profiles. Finally, more general velocity propagators can be considered with weaker singularities (wave-like parts).
(d) Intermediate “leakage” models, energy transport and the self-consistency of the turbulence/wave FIF models

We have presented a model in which the space-time propagator corresponds to a fractional integral over waves with a nonlinear turbulent dispersion defined via the physical scale function. The implications for energy transport are very strong: the turbulent energy flux input will generally be transported far from the source via the waves. Since the energy flux is related to the velocity shears via $\varepsilon=\Delta v^3/\Delta x$, most of the energy must remain localized for the model to be self-consistent. One way of achieving this energy localization would be if the dispersion relation had a negative imaginary part. However this would imply a dissipation mechanism which – if too strong – would contradict the picture of a cascade of conservative fluxes upon which the FIF is built. A more satisfying method is to combine the wave with the turbulent scale functions so that the final model has aspects of both; in this case, the wave energy could considered as “leakage” in analogy with the Lumley-Shur model. A simple way to achieve this is to use:

$$g_\nu(k,\omega) = \left[h(t)\mathbb{1}_{[r,t]}(k-\alpha_D)^{-1/H}\frac{1}{2}ight]^{-1/H} \cdot H_{\text{wav}}; \quad H_{\text{tur}} + H_{\text{wav}} = H$$

We see that the extreme localized and extreme unlocalized models correspond to $H_{\text{tur}}=H$, $H_{\text{wav}}=0$ and $H_{\text{tur}}=0$, $H_{\text{wav}}=H$ respectively (recall $H=1/3$). In Fig. 8 we show the effect of increasing $H_{\text{wav}}$: one can see how structures become progressively more and more wave-like while retaining the same scaling symmetries, close to observations.
Figure 8. This figure shows the effect of increasing $H_{\text{wav}}$ with $H_{\text{wav}}+H_{\text{tur}}=1/3$, $H_t=2/3$; clockwise from the upper left we have $H_{\text{wav}} = 0, 0.33, 0.52, 0.38$ (i.e. $H_{\text{tur}}=1/3-H_{\text{wav}}=0.33$, 0, -0.19, -0.05), $C_f=0.1$, $\alpha=1.8$. There is a small amount of differential anisotropy characterized by $G = \begin{pmatrix} 0.95 & -0.02 \\ 0.02 & 1.05 \end{pmatrix}$. The horizontal unit ball is characterized by $\Theta(\Omega) = 1 + a \cos(2\Omega - 2\Omega_0)$ with $a=0.65$, with $\Omega_0 = 0$. The random seed is the same in all cases so that one can see how structures become progressively more and more wave-like while retaining the same scaling symmetries, close to observations.
4. CONCLUSIONS

One of the most fundamental unsolved problems in atmospheric science is to understand the nature of the stratification over wide ranges of space-time scales. Since the 1980’s, the classical isotropic 3D/isotropic 2D model has been abandoned in the face of mounting empirical evidence that the vertical and horizontal scalings are different. This lead to the development of several theories based on linear gravity waves and a competing strongly nonlinear turbulence flux based on 23/9D model in which the horizontal dynamics are controlled by energy fluxes, and the vertical by buoyancy force variance fluxes. This model is based on the notion of physical scale – that the nonlinear turbulent dynamics determines the physically relevant scale, we do not impose a priori classical (Euclidean) scales. The stochastic implementation of the 23/9D model; the “Fractionally integrated Flux” (FIF) model is more satisfying because it assumes strongly nonlinear turbulence and empirically atmospheric Reynolds numbers are of the order $10^{12}$. Although the FIF model has few restrictions, initial implementations focused on the special case in which structures were localized in both space and in space-time. This implied that they lacked wave phenomenologies; numerical multifractal modeling confirmed this feature.

The key result in this paper is to show that the FIF framework is wide enough not only to include the wave effects, but even dispersion relations sufficiently close to the standard gravity wave dispersion relations that the model can quite plausibly explain the empirical results in much of the atmospheric gravity wave literature. The key point is that the FIF requires two propagators (space-time Green’s functions). The first determines the space-time structure of the cascade of fluxes, this must be localized in space-time in order to satisfy the usual turbulence phenomenology. In contrast, the second propagator relates the turbulent fluxes to the observables, this propagator can still be localized in space but can be unlocalized in space-time (the spatial part is the same as before, it is spatially localized in wave packets).

This model is still very general – it’s main constraint is to produce stochastic realizations which respect the anistotropic, multifractal extensions of Kolmogorov’s law (or in the case of passive scalars, Corrsin – Obukhov laws). The turbulence determined “physical” scale function defines an anisotropic dispersion relation; by changing the scale function we can change the dispersion relation. In order to allow the model to account for the numerous observations of gravity waves, we show how a specific “gravito-turbulence” dispersion relation can be chosen which has most of the key qualitative features of the classical linear dispersion relation. We show by numerical simulations that the new model does indeed generate wave-like phenomenologies. Finally, we show how to interpolate between the extreme wave and turbulence propagators so that any degree of localization/delocalization can be accommodated. In parts II, III, we use high-powered lidar data of passive scalars to accurately confirm the overall FIF framework; however the observations are still not accurate enough to distinguish the various specific localized/unlocalized FIF models discussed here. This means that theory and numerical multifractal modeling will be particularly important.
5. REFERENCES


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