Towards a new synthesis for atmospheric dynamics: space-time cascades

S. Lovejoy\textsuperscript{1,2}, D. Schertzer\textsuperscript{3,4}

\textsuperscript{1}Physics, McGill University, 3600 University st., Montreal, Que., Canada
\textsuperscript{2}GEOTOP, UQAM, CP 8888, Succ. Centre Ville, Montreal, Que., H3C 3P8, Canada
\textsuperscript{3}Université Paris-Est, ENPC/CEREVE, 77455, Marne-la-Vallée Cedex 2, France
\textsuperscript{4}Météo France, 1 Quai Branly, 75007 Paris

Abstract:

In spite of the unprecedented quantity and quality of meteorological data and numerical models, there is still no consensus about the atmosphere’s elementary statistical properties as functions of scale in either time or in space. This review paper proposes a new synthesis based on a) advances in the last 25 years in nonlinear dynamics, b) a critical re-analysis of empirical aircraft and vertical sonde data, c) the systematic scale by scale, space-time exploitation of high resolution remotely sensed data and d) the systematic re-analysis of the outputs of numerical models of the atmosphere, e) a new turbulent model for the emergence of the climate from “weather” and climate variability. We conclude that Richardson’s old idea of scale by scale simplicity – today embodied in multiplicative cascades – can accurately explain the statistical properties of the atmosphere and its models over most of the meteorologically significant range of scales, as well as at least some of the climate range. The resulting space-time cascade model combines these nonlinear developments with modern statistical analyses, it is based on strongly anisotropic and intermittent, generalizations of the classical turbulence laws of Kolmogorov, Corrsin, Obukhov, and Bolgiano.

1. Introduction:

1.1 The golden age paradox:

As little as 25 years ago, few atmospheric data sets spanned more than two orders of magnitude in scale; yet they were challenging even to visualize. Global models had even lower resolutions, yet required heroic computer effort. The atmosphere was seen through a low resolution lens. Today, in situ and remote data routinely span scale ratios of $10^3$ to $10^4$ in space and/or time scales and operational models are not far behind. We are now beginning to perceive the true complexity of atmospheric fields which span ratios of over $10^{10}$ in spatial scales (the planet scale to the dissipation scale).

Increasingly we are faced with the problem of achieving a scale by scale understanding of complex hierarchies of structures embedded within structures. One (still) popular approach is through phenomenological models based on subjectively defined morphologies or other ad hoc features; in some schools new models and mechanisms are introduced every factor of 2 or so in scale. A more theoretically
satisfying approach is via statistical physics, i.e. turbulence theory. The still “standard”
turbulence model was elaborated at the end of the 1960’s (see e.g. (Monin, 1972)’s
influential book); it was critically based on the theory of two dimensional isotropic
turbulence (Kraichnan, 1967), (Leith, 1968) with (Charney, 1971)’s extension to
geostrophic turbulence. This classical picture of the atmosphere consists of a 3D
isotropic turbulence at small scales and a 2D isotropic turbulence at large scales with the
two separated by a “meso-scale gap” somewhere near the 10 km scale thickness. In
addition to the scale at which this “dimensional transition” supposedly occurs, in
principle two additional larger scales are needed: one for the 2-D energy (velocity
variance) flux injection, another for the enstrophy (vorticity variance) flux injection.
With all these scales to play with, it is easy to concoct complicated models. The more
realistic proposals involve i) a small scale isotropic 3D turbulence regime with energy
cascading to smaller scales, ii) an intermediate downscale 2-D isotropic enstrophy
(vorticity variance flux) cascade regime and iii) a large scale, 2-D isotropic upscale
ergy cascade regime (e.g. (Lilly, 1983), (Gage and Nastrom, 1986), (Lilly, 1989),
(Gifford, 1988), (Bartello, 1995)).

One might have expected that with today’s quantity and quality of satellite,
aircraft and other remote and in situ measurements, that this basic picture of the
atmosphere would by now have been either accepted and then refined or rejected and
then replaced, but this has not been the case; indeed, the status of this hierarchy of
isotropic turbulence models is still under debate today. On the one hand, empirically the
situation in the horizontal is still contradictory (see the discussion below) on the other
hand, theoretically the finding that the 3D regime may destabilize the 2D regime
(Bartello, 1995) raises issues about its internal consistency (see e.g. (Hamilton et al.,
2008) for a recent numerical investigation).

The situation in the horizontal is bad enough, but when we consider the vertical
structure of the atmosphere, it is even more embarrassing. This is because vertical
spectra in three dimensional isotropic turbulence are totally different from those in 2D
turbulence so that the violations of the predictions of the classical 2D/3D picture are all
the more flagrant. Indeed, starting in the 1980’s experiments claimed that the turbulence
was anisotropic ((Van Zandt, 1982), (Schtzer and Lovejoy, 1985b)), and during the
1990’s for the key horizontal wind, the mainstream experimental community favoured a
$k_z^{-3}$ model for the vertical spectrum along with a $k_x^{-5/3}$ horizontal spectrum, and this out
to scales $>>$ 10 km ($k_z$, $k_x$ are vertical and horizontal wavenumbers respectively), i.e. a
fundamentally anisotropic model. The most popular models reproducing this behaviour
were the Saturated Wave Theory (SWT, (Dewan, 1997)), and the Diffusive Filtering
Theory (DFT, (Gardner, 1994)). Since both the SWT and DFT are based on (quasi)
linear gravity waves neither are turbulent at all! The divorce between the dominant
strongly nonlinear isotropic theories and the dominant weakly nonlinear and anisotropic
interpretations of the experimental data is almost total.

1.2 Is the atmosphere irreducibly complex?

These inconclusive attempts to understand the scale by scale structure of the
atmosphere have reinforced the idea that the atmosphere is irreducibly complex, that it is
only amenable to brute force numerics. Yet, the history of science is replete with
examples showing that apparent complexity can hide simplicity at another level. In the atmosphere it is ironic that the possibility of scale by scale simplicity – through cascade processes – was first posed by precisely the pioneer of the brute force approach, Lewis F. Richardson (Richardson, 1922) (see fig. 1 for a modern schematic of a cascade).

This paper is an overview of recent work addressing this brute force complexity / scale by scale simplicity dichotomy i.e. the cascade alternative. This review is timely for several reasons. The first is that the development of a multiscale synthesis based on an anisotropic cascade alternative to the dominant 2D isotropic/ 3D isotropic picture has reached a point where - profiting from the golden age - it has amassed a large and compelling body of empirical evidence in its favour. The second is that it can easily and naturally explain the most comprehensive existing empirical studies - and this without resorting to ad hoc/ speculative models or reasoning. The third is that recent studies of large scale numerical models of the atmosphere show that these models are themselves accurately described by multiplicative cascade process (Stolle et al., 2009) so that if we reject cascades, then we also reject the most advanced and prestigious numerical models. The final reason is that the stochastic cascade approach is increasingly needed for applications. This is particularly true since the last twenty years has witnessed a revolution in numerical weather prediction. Twenty years ago the goal of weather forecasting was to determine the (supposedly unique) state of the atmosphere at some time in the future, whereas today, ensemble forecasting systems have instead the goal of determining the possible states of tomorrow’s weather including their probabilities of occurrence. This new goal corresponds to a transition from deterministic to stochastic forecasts. Today’s ensemble forecasting systems therefore require knowledge of the underlying stochastic structure of the deterministic equations, knowledge that the cascades conveniently provide.

1.3 The alternative: multiplicative cascades

Figure 1 shows the basic cascade idea: due to instability and / or interactions with other structures, an “eddy” breaks up into “daughter” eddies passing on (conserved) fluxes to the smaller structures. These phenomenological models are based on three basic aspects of turbulent fluid dynamics; a) the existence of a quantity conserved by the nonlinear terms in the equations, b) the existence of a wide range without characteristic scales, c) the fact that interactions are strongest between structures of neighbouring size: “locality in fourier space”. Some comments are needed. First, the conserved quantity is usually taken as the energy and/or enstrophy fluxes; sometimes potential enstrophy flux, but the cascade notion is more general. Second, the notion of a scaling range is usually reduced to that of an “inertial range”, i.e. a range without sources or sinks of flux, but this is also unnecessarily restrictive: it is sufficient that the sources and sinks are themselves scaling, hence the significance of the cascade study of the short and long wave radiances (section 2, (Lovejoy et al., 2009a)). In other words, the scaling symmetries are taken as the basis. Third, while 3D turbulence is local in Fourier space, 2D turbulence is not, so that the finding of large scale multiplicative cascades is prima facie evidence against the relevance of 2D turbulence models to the atmosphere.

In much of the atmospheric literature, cascades have been invoked at a somewhat vague conceptual level. In conjunction with dimensional arguments based on the scale by scale conserved turbulent flux (primarily energy flux, ε enstrophy flux, η)
they are nevertheless able to determine the basic (nonintermittent) spectra (the classical $k^{-5/3}$, $k^3$ respectively) and with the help of statistical mechanical arguments, they can determine the cascade direction (downscale for energy flux in 3D and enstrophy flux in 2D, upscale for energy flux in 2D), see e.g. (Chen and Wiin-Nielsen, 1978), (Boer and Shepherd, 1983), (Bartello, 1995), (Strauss and Ditlevsen, 1999). However, to progress further requires the development of explicit, multiplicative cascade models ((Novikov and Stewart, 1964), (Yaglom, 1966), (Mandelbrot, 1974)). To see how these models work, consider fig. 2 a which shows the development of a 1-D, 2-state discrete in scale cascade developed with discrete cascade ratio $\lambda_0 = 2$. The unit interval is divided into $\lambda_0$ parts and then a random multiplier $\mu\varepsilon$ is chosen for each half (each daughter). In the 2-state process shown, the multiplier is one of two possible values $\mu\varepsilon = \lambda_0^{\gamma^+}$ or $\mu\varepsilon = \lambda_0^{\gamma^-}$ with $\gamma^+ > 0$, $\gamma^- < 0$ corresponding to boosts or decreases respectively. To enforce scale by scale conservation of the flux, we require the average multiplier to be normalized. Using ensemble (“canonical”) conservation, $<\mu\varepsilon> = 1$ we obtain the “$\alpha$ model” (Schertzer and Lovejoy, 1985b). If the much stronger “microcanonical” conservation $(\mu\varepsilon_{\text{left}} + \mu\varepsilon_{\text{right}})/2 = 1$ is used, the result is the “$\rho$ model” (Meneveau and Sreenivasan, 1987). Due to the strong limitations that microcanonical conservation implies, they are unrealistically “calm”, and we will not discuss microcanonical models further. The process of division and random multiplication is then continued to smaller and smaller scales, this limit is discussed below, see fig 2 b for a 2D example.
Fig. 1: The left of the figure shows a schematic of an isotropic cascade. Due to nonlinear interactions with other eddies or due to into instability, a large eddy/structure (indicated as a square) breaks up into daughter eddies (smaller squares). Following the left most arrow the energy flux is redistributed uniformly in space, the result is a homogeneous (non fractal) cascade. Following the right hand arrow, at each cascade step, we randomly allow one eddy in four to be “dead”, the result is that turbulence is only active on a fractal set. At the bottom, we see the average shape as a function of scale of more realistic (isotropic eddies). The right hand column is the same except that it shows an anisotropic cascade, a model of a vertical cross-section of the atmosphere.
(on the left a homogeneous, on the right, inhomogeneous, fractal model). The degree of stratification is characterized by an elliptical dimension $D_{el} = 1.5$ in the example. Adapted from (Schertzer and Lovejoy, 1987).

Fig. 2a: A schematic of a two-state (binomial, “α model”) cascade in 1-D showing the first step in the cascade with randomly chosen boosts by factor $\lambda^+$ and decreases by factor $\lambda^-$. 
Fig. 2: 2D extension of the $\alpha$ model shown in fig. 2 a (figure from (Wilson et al., 1991)). As we move from top to bottom on the left side, more and more cascade steps are added. We see that since on average the integral under the spikes is conserved so that while the strength of the spikes tends to increase at each step, this effect is offset by low intensities (low $\varepsilon$) throughout most of the region. On the right we indicate the “dressed” cascade obtained by integrating the completed “bare” cascade over the same scales. Note that occasionally the spikes on the right are much higher, due to the small scale activity which is not fully “averaged out” and is associated with the divergence of statistical moments, self-organized critical behaviour.

By the mid 1980’s, it was recognized that these multiplicative cascade models were quite general, leading to the following generic statistical behaviour:

$$\langle \epsilon^q \rangle = \lambda^{K(q)}$$

(1)
where $\lambda \geq 1$ is the overall ratio of scales (largest over smallest; $\lambda = \lambda_0^n$ for $n$ steps of a discrete in scale model) over which the cascade has developed and $\varepsilon_{\lambda}$ is the scale by scale conserved flux. $K(q)$ is the moment scaling function which contains the statistical information about the process. At this very general level, $K(q)$ need only satisfy certain loose requirements, principally that it is convex and – due to the scale by scale conservation $<\varepsilon_{\lambda}> = 1$; $K(1) = 0$. However an appropriate multiplicative extension of the central limit theorem shows that under fairly general circumstances it is of the “universal multifractal” type (Schertzer and Lovejoy, 1987), (Schertzer and Lovejoy, 1997) characterized by two basic parameters:

$$K(q) = \frac{C_1}{\alpha - 1} \left( q^\alpha - q \right)$$

(2)

where $0 \leq C_1 \leq D$ is the “codimension of the mean” ($D$ is the dimension of the space over which the cascade is developed) and $0 \leq \alpha \leq 2$ is the “index of multifractality” characterizing the degree of multifractality: $\alpha = 0$ is the monofractal limit, $\alpha = 2$ the “lognormal” maximum. $\alpha$ is also the Levy index of the generator ($\log \varepsilon$; see section 4.4).

So far we have described the “bare” cascade quantities, i.e. the statistical properties of a cascade developed over the finite scale ratio $\lambda$. However, it is important to realize that the small scale cascade limit ($\lambda \to \infty$) is highly singular, indeed, from eq. 1 we see that the second characteristic functions of $\log \varepsilon$ (i.e. $K(q) \log \lambda$) diverge; it turns out that meaningful limiting properties only exist for integrals of the cascade over finite sets (this is analogous to the Dirac $\delta$ function which is defined as a limit of functions and which is only meaningful under an integral sign). If these integrals are over sets at scale $\lambda$, then the resulting “dressed” processes (shown as the right hand column in fig. 2 b) have long probability tails such that all moments $q > q_D$ diverge i.e. $Pr(\varepsilon_{\lambda} > s) \approx s^{-qD}$ for $s >> 1$ where $q_D$ is a critical order of divergence depending on both the statistical properties ($K(q)$) and the dimension $D$ of the space in which the process is averaged/observed. There have indeed been many claims of such “self-organized critical” ((Bak et al., 1987)) behaviour in atmospheric fields ranging from the wind, temperature and rain ((Schertzer and Lovejoy, 1985b), (Schertzer and Lovejoy, 1987), (Lovejoy and Schertzer, 1995), (Tchiguirinskaia et al., 2006), (Lovejoy and Schertzer, 2007).

The cascades described above and the example in fig. 2 are discrete in scale; they only exactly obey eq. 1 for integer powers ($n$) of integer ratios: $\lambda = \lambda_0^n$. In themselves they are academic; real processes are presumably continuous in scale (i.e. they respect eq. 1 for any $\lambda$ within the wide scaling range). In spite of this shortcoming, they nevertheless display all the basic features of more realistic continuous in scale (sometimes called “infinitely divisible”) models, in physics jargon they are called “toy models”. Fig. 3 a, b, c, d shows examples of more realistic continuous in scale cascades.

This paper is structured as follows. In section 2 we give some basic theory (2.1), we then discuss horizontal analyses of models (2.2), remotely sensed radiances (2.3), atmospheric boundary conditions (2.4), aircraft measurements (2.5). In section 3 we turn our attention to the vertical, first discussing some theory of anisotropic
scaling/stratification (3.1), implications for aircraft measurements (3.2), lidar measurements of atmospheric cross sections (3.3), drop sondes (3.4), and an intercomparison of the stratification of different atmospheric fields. In section 4 we consider the extension of spatial cascades to the time domain; section 4.1 is a discussion, 4.2 gives some theory, 4.3 some evidence of space-time scaling from lidar, section (4.4) from satellites and numerical models. In 4.5 we discuss the extension of the model to climate scales arguing that climate may be a dimensional transition (still) multiplicative process. In section 5 we conclude.

Fig. 3 a: Examples of continuous in scale anisotropic multifractals in 3D. The effect of changing the spherically-scale (section 3) on multifractal models of clouds. Only single scattering is used for the radiative transfer rendition. The cloud parameters are: $\alpha = 1.8$, $C_1 = 0.1$, $H = 1/3$, $r =$ vertical/ horizontal aspect ratio top to bottom : $r = 1/4, 1, 4$, left to right, spherically-scale $= 1, 8$. For technical details, see (Lovejoy and Schertzer, 2008).
Fig. 3 b: The same clouds, but a side view showing how the stratification at each fixed scale changes with changing spheroc scale. However, the exponent \((H_z, D_{el})\) characterizing the differential (scale by scale) stratification is the same.
Fig. 3c: The bottom row shows multifractal simulations of topography; the parameters $(H = 0.7, C_1 = 0.12, \alpha = 1.8)$ are those empirically determined (as discussed in section 2.4); the only difference is the type of scaling anisotropy. The top row shows the corresponding scale functions which define the notion of scale (adapted from (Gagnon et al., 2006)). The left is a self-similar model (isotropic), the middle is linear GSI with generator $G = ((0.8, -0.05),(0.05,1.2))$ and with a canonical scale function, the right has the same $G$ as the middle but a more complicated unit ball.
Fig. 3d: This figure shows a model using scale functions respecting the anisotropic (vertically stratified) extension of the Corrsin-Obukhov scaling laws but which are localized only in sspace, not in space-time (i.e. they are wave-like). The degree of space-time delocalization increases clockwise from the upper left. The parameters are, $H = 0.33$, $C_1=0.1$, $\alpha = 1.8$. There is a small amount of differential anisotropy characterized by $G = \begin{pmatrix} 0.95 & -0.02 \\ 0.02 & 1.05 \end{pmatrix}$. The random seed is the same in all cases so that one can see how structures become progressively more and more wave-like while retaining the same scaling symmetries, close to observations.

2. Multiplicative cascades in the horizontal

2.1 Discussion:

The ideal test of multiplicative cascades would be to have large quantities of dynamical wind data in both the horizontal and vertical, preferably on regular grids at fine resolution - or failing that - of other high quality in situ measurements. But in situ data are not what they seem: in situ measurements are typically spatially clustered over wide ranges of scale, the networks typically turn out to be fractal ((Lovejoy et al., 1986), (Korvin et al., 1990), (Nicolis, 1993), (Doswell III and Lasher-Trapp, 1997; Mazzarella and Tranfaglia, 2000), (Giordano et al., 2006)) and require statistical corrections ((Tessier et al., 1994)). Similarly, aircraft trajectories are also fractal and require a theory of turbulence (isotropic / anisotropic) to interpret them ((Lovejoy et al., 2004), (Lovejoy et al., 2009d) and see section 2.5 below). Finally the state of the art vertical analysis device - drop sondes - turn out to have multifractal outages and require special analysis techniques ((Lovejoy et al., 2009c)) followed by statistical corrections for biases. Traditionally sophisticated data assimilation techniques have been used to attempt to overcome these difficulties but this at the price of introducing a number of ad hoc regularity and smoothness assumptions.

Another obstacle to testing the predictions of cascade models is that the atmosphere is “thin”: its scale height is $\approx 10$ km so for the cascades to operate over significant horizontal ranges, they must be stratified, anisotropic: we need a theory of anisotropic turbulence (section 3). It turns out that at least with regard to aircraft data the two difficulties are linked: a theory of anisotropic turbulence is needed simply in order to properly interpret the data (section 3.2).

Alternatively, we can take advantage of the fact that the different atmospheric fields are strongly nonlinearly coupled over wide ranges of scale. From a cascade perspective, this implies a model involving several interacting cascades (one for each conserved flux; this is a generalization of the classical Corrsin – Obhukov theory for passive scalar advection, (Corrsin, 1951), (Obukhov, 1949)). Due to these couplings over wide ranges of scale, we fairly generally expect atmospheric fields - including remotely sensed radiances - to display cascade structures. A straightforward way to progress is therefore to exploit the large quantities of remotely sensed data, we discuss this in section 2.3; we start however with an analysis of the output of numerical models which are particularly easy to interpret.
2.2 Testing the predictions multiplicative cascades:

We have outlined a model of the atmosphere consisting of interacting cascades, each corresponding to a different scale by scale conserved turbulent flux. In order to test the model, we must have some way to determine the fluxes from the observations. The usual approach has been to use isotropic (2D, 3D) turbulence theories to decide \textit{a priori} the appropriate flux (typically the energy or enstrophy flux) and to predict the spectral exponent of wind, temperature and other observables accordingly. To date, the most ambitious attempt in this direction is that by (Strauss and Ditlevsen, 1999) who followed the classical (Boer and Shepherd, 1983) methodology of applying 2D turbulence theory to reanalyses. Recall that reanalyses are the result of applying modern data assimilation techniques to meteorological data. Accordingly, starting with reanalyses, (Strauss and Ditlevsen, 1999) first vertically integrated the wind field to reduce the problem from 3D to 2D, they then separated it into mean and transient, rotational and irrotational components, determining the spectra of the resulting highly processed fields (fig. 4a,b, T106, 31 levels, 15 year data set, 1979-1993). Their conclusions were unequivocal: the reanalyses did not show the key signatures expected of 2D turbulence over any range – there was no credible $k^{-3}$ spectral range nor was there a significant upscale transfer of energy. While the former would be indicative of an enstrophy cascade dominated range, the latter would be indicative a large scale energy cascade regime (both are predicted in 2D turbulence).
Fig. 4a: The spectra of the wind transients. The spectra are in pairs of winter and summer results; the bottom pair are divergent the top pair rotational. The reference lines (added) have absolute slopes 5/3, 3. Adapted from (Strauss and Ditlevsen, 1999).

Fig. 4b: The spectra of the means: The spectra are in pairs of winter and summer results; the bottom pair are divergent the top pair rotational. The reference lines (added) have absolute slopes 5/3, 3. Adapted from (Strauss and Ditlevsen, 1999).

In order to test eq. 1, we must therefore use an approach that does not require a priori assumptions about the physical nature of the relevant fluxes; nor of their scale symmetries (isotropic or otherwise) the latter will in fact turn to out to be anisotropic and hence nonclassical. If atmospheric dynamics are controlled by scale invariant turbulent cascades of various (scale by scale) conserved fluxes \( \phi \) then in a scaling regime, the fluctuations \( \Delta I(\Delta x) \) in an observable (e.g. wind, temperature or radiance) over a distance \( \Delta x \) are related to the turbulent fluxes by a relation of the form:

\[
\Delta I(\Delta x) \approx \phi \Delta x^H
\]  

(3) a

(this is a generalization of the Kolmogorov law for velocity fluctuations, the latter has \( H = 1/3 \) and \( \phi = \epsilon^\eta \), \( \eta = 1/3 \) where \( \epsilon \) is the energy flux to smaller scales). Without knowing \( \eta \) or \( H \) - nor even the physical nature of the flux - we can use this to estimate the normalized (nondimensional) flux at the smallest resolution (\( \Delta x \)) of our data:

\[
\frac{\phi}{\langle \phi \rangle} = \frac{\Delta I}{\langle \Delta I \rangle}
\]

(3) b
where “<>” indicates statistical averaging. Note that if the fluxes are realizations of pure multiplicative cascades then the normalized flux power $\eta$ is also pure multiplicative cascades, so that $\varphi / \langle \varphi \rangle$ is a normalized cascade. The fluctuation, $\Delta I(\Delta x)$ can be estimated in various ways; in 1-D a convenient method (which works for $0 \leq H \leq 1$) is to use absolute differences: $\Delta I(\Delta x) = |I(x + \Delta x) - I(x)|$ with $\Delta x$ the smallest reliable resolution where $x$ is a horizontal coordinate, but alternatively other definitions of fluctuations could be used (this is sometimes called “the poor man’s wavelet”; other wavelets could be used). In 2-D, convenient definitions of fluctuations are the (finite difference) Laplacian (estimated as the difference between the value at a grid point and the average of its neighbours), or the modulus of a finite difference estimate of the gradient vector. The resulting high resolution flux estimates can then be degraded (by averaging) to a lower resolution $L$.

Following eq. 1, the basic prediction of multiplicative cascades applied to a turbulent flux is that the normalized moments $M_q = \langle \varphi_x^q \rangle / \langle \varphi_x \rangle^q$ obey the generic multiscaling relation:

$$M_q = \left( \frac{\lambda}{\lambda_{\text{eff}}} \right)^{K(q)} ; \quad \lambda = L_{\text{earth}} / L ; \quad \lambda_{\text{eff}} = L_{\text{earth}} / L_{\text{eff}}$$

where “<>” indicates statistical (ensemble) averaging and $L_{\text{eff}}$ is the effective outer scale of the cascade. $\langle \varphi_x \rangle$ is the ensemble mean large scale (i.e. the climatological value). $\lambda$ is a convenient scale ratio based on the largest great circle distance on the earth: $L_{\text{earth}} = 20,000$ km and the scale ratio $\lambda / \lambda_{\text{eff}}$ is the overall ratio from the scale where the cascade started to the resolution scale $L$, it is determined empirically.

Since empirical data are nearly always sampled at scales much larger than the dissipation scales, the above scaling range based technique for estimating fluxes has general applicability. If instead we have dissipation range data (for example if we estimate fluxes from the outputs of numerical models at the model dissipation scale), then the basic approach still works, but the interpretation is a little different. To see this, we continue with the example of the energy flux, recalling that at the dissipation scale:

$$\mathcal{E} \approx v^2 \cdot \nabla^2 v$$

where $v$ is the viscosity, $\nabla$ the velocity. Standard manipulations give:

$$\mathcal{E} \approx v \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 \approx v \left( \frac{\partial v}{\partial x} \right)^2$$

so that if $\Delta x$ is in the dissipation range (e.g. the finest resolution of the model) then:

$$\Delta v \approx \left( \frac{\mathcal{E}}{v} \right)^{1/2} \Delta x$$

Since the models actually use hyper-viscosities with hyper viscous coefficient $v^*$ and Laplacian exponent $h$ (typically $h = 3$ or 4), we have:

$$\mathcal{E} \approx v^* \cdot \nabla^{2h}$$

$$\Delta v \approx \left( \frac{\mathcal{E}}{v^*} \right)^{1/2} \Delta x$$
which leads to the estimate:

\[ \Delta v \approx \left( \frac{\varepsilon}{V^2} \right)^{1/2} \Delta x^h \]  

(9)

In all cases, we therefore have (independent of \( h \)):

\[ \frac{\Delta v}{\langle \Delta v \rangle} = \frac{\varepsilon^{1/2}}{\langle \varepsilon^{1/2} \rangle} \]  

(10)

We see that this is the same as equation 3 b, the only difference is that for the wind field, the exponent \( \eta = 1/2 \) holds in the dissipation range rather than \( \eta = 1/3 \) which holds in the scaling regime. If we introduce \( Kn(q) \) which is the scaling exponent for the normalized \( \eta \) flux \( \phi/\langle \phi \rangle = \varepsilon^\eta/\langle \varepsilon^h \rangle \), then \( Kn(q) = K(q) - qK(\eta) \) which for universal multifractals yields \( Kn(q) = \eta^q K(q) \) i.e. in obvious notation, \( C_{1,\eta} = \eta^\alpha C_{1,1} \) (c.f. eq. 2) so that comparing the dissipation estimate (\( \eta = 1/2 \)) and the scaling range estimate (\( \eta = 1/3 \)), we have:

\[ C_{1,\text{diss}} = \left( \frac{3}{2} \right)^\alpha C_{1,\text{scaling}} \]  

(11)

Since we find (for the wind field) that \( \alpha \approx 1.8 \) we have \( C_{1,\text{diss}}/C_{1,\text{scaling}} \approx 1.5^{1.8} \approx 2.07 \).

The extension of this discussion to passive scalars is also relevant and shows that the interpretation of the empirically/numerically estimated fluxes in terms of classical theoretical fluxes can be nontrivial. Denoting by \( \rho \) the density of the passive scalar, and \( \chi \) its variance flux, the dissipation range formula analogous to eq. 5 is \( \chi \approx \rho \kappa^2 \alpha \) (\( \kappa \) is the molecular diffusivity, assumed constant) leading to \( \Delta \rho = (\chi / \kappa)^{1/2} \Delta x \) whereas the corresponding formula in the scaling range is \( \Delta \rho \approx \chi^{1/2} \varepsilon^{-1/6} \Delta x^{1/3} \) (the Corrsin Obukhov law) which has the same dependency on \( \chi \), but which also involves the energy flux; the combined effective flux \( \phi = \chi^{1/2} \varepsilon^{-1/6} \) measured by the scaling method thus involves two (presumably statistically dependent) cascade quantities rather than just one. In summary, although both dissipation and scaling ranges can be used to test for multiplicative cascades and to quantify their variability, the relation between the two normalized fluxes is not necessarily trivial.

We can now return to the reanalyses mentioned above; we summarize the salient points of (Stolle et al., 2009) who used three years of the ERA40 project reanalyses based on the ECMWF model ((Uppala, 2005)); these are somewhat higher resolution versions of the reanalyses studied by (Strauss and Ditlevsen, 1999). Recall that a reanalysis is by no means a pure empirical field, it is rather a highly elaborated “product” obtained in this case by using sophisticated 4-D variational data assimilation techniques, themselves are based on various smoothness and regularity assumptions (i.e. they don’t take the strong resolution dependencies - eq. 1 - into account, they assume that at scales smaller than one pixel \( K(q) = 0 \)).

Fig. 5a, b, c shows the result for the lowest level (1000 mb) horizontal wind temperature and humidity fields respectively (due to topography, the interpretation of the global 1000 mb field is not without problems; however the results at higher levels were qualitatively and quantitatively very similar, see (Stolle et al., 2009) and table 1). From the existence of converging straight lines, we see that the predictions of the cascade model are accurately obeyed (we quantify this below). Indeed what is
particularly striking is that not only are the outer scales near those of the planet, but in addition that they are accurately followed up to at least $\approx 5000$ km so that eq. 1 holds over nearly the full available range. This is perhaps surprising since one might have expected a larger range of scales to be required before this “asymptotic” cascade structure is attained at smaller scales.

In order to quantify the accuracy to which this cascade scaling is obeyed, we can determine the small deviations by estimating the mean absolute residuals:

$$\Delta = \log_{10}\left(\frac{M_q}{K(q)\lambda_{\text{eff}}^{q}}\right)$$

For each $q$, $\Delta$ is determined from the linear regression on Fig. 5; the slopes yield $K(q)$ and $\lambda_{\text{eff}}$ is determined from the intercept (fixed to be the same for all $q$). The overbar in equation (5) indicates averaging over the different $\lambda$ (at intervals of $10^{0.2}$) over the available range of scales up to 5000 km and over the moments $0 \leq q \leq 2$ (corresponding to $> 90\%$ of the data). $\delta$ is the percentage deviation; we find $\delta \approx 0.5\%$, the details are given in Table 1a, b along with estimates of the outer scales and the corresponding characterization by universal multifractals (i.e. the parameters $C_1, \alpha$ in eq. 2).

From the table, we see that very similar results were found for forecasts of a Canadian global weather model GEM (fig. 5 b, d, f) and the National Weather Service GFS model; for example the deviations are of the order $\pm 0.3\%$ for the reanalyses, $\pm 0.5\%$ for GEM and $\pm 0.5\%$ for GFS (table 1). These small deviations allow us to conclude that the analyses and models accurately have a cascade structure. Overall, from the table we can also see that the $K(q)$ “shape parameter” - the difficult to estimate multifractal index $\alpha$ - is roughly constant at $\alpha = 1.8 \pm 0.1$. From table 1 a, we see that the scale by scale characterization of the intermittency near the mean ($C_1$) has a tendency to decrease with altitude, this being somewhat amplified by a decrease in the external scale (which decreases all the moments by the same factor). Interestingly, the $C_1$ is very similar for the different fields (it is slightly larger for the humidity), although as expected from our discussion of the difference between dissipation and scaling range flux estimates the $C_1$ are quite a bit larger than those measured by aircraft (section 2.5), also shown in the table, the difference is roughly the factor of 2 estimated in eq. 11 for the velocity field (i.e. the dissipation versus the scaling range flux estimate).

In table 1 b, we compare the two forecast models (GEM, GFS) in order to see if there are any systematic trends as the model integrations increase (the effect of initial conditions becomes less and less important). No systematic trends are obvious, although for the 144 hour GFS forecast, the scaling is notably poorer (although still excellent) with deviations of up to $\pm 1.5\%$. Note that because even the longest available forecast is still statistically influenced by the analyses, these results do not (quite) establish that the long time behaviour of the model is cascade-like. Below, we examine the cascade behaviour in the time domain.
Fig. 5: Moments of fields at 1000 mb for $q = 0.0$ to 2.9 in steps of 0.1. The $x$ axis is $\log_{10} \lambda$, $\lambda = L_{\text{earth}}/L$, $L_{\text{earth}} = 20,000$ km. The left column is the ERA40 reanalyses, the right column is the GEM analysis ($t = 0$). The rows top to bottom are temperature, east-west wind, humidity. For the parameters, see table 1a, b. From Stolle et al., 2009.
Table 1a: Intercomparison of initial ($t = 0$) fields for various fields at 1000, 700 mb. The triplets of values are for, ERA40, GEM, GFS respectively. The aircraft estimates are from section 2.5 and are at about 200 mb (the figure in parentheses is the section 2.5 result, the second is corrected by the factor $(3/2)^{1.8} = 2.07$ needed – at least for the wind field - to estimate the dissipation scale $C_1$ from the scaling range $C_1$, see eq. 11).

<table>
<thead>
<tr>
<th>Field</th>
<th>$C_1$</th>
<th>$\alpha$</th>
<th>$L_{\text{eff}}$ (km)</th>
<th>$\delta$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$ (1000)</td>
<td>0.113, 0.125, 0.142</td>
<td>1.94, 1.64, 1.72</td>
<td>21900, 25800, 28000</td>
<td>0.31, 0.27, 0.59</td>
</tr>
<tr>
<td>$T$ (700)</td>
<td>0.094, 0.077, 0.080</td>
<td>2.11, 1.94, 2.00</td>
<td>14500, 8300, 8600</td>
<td>0.29, 0.47, 1.02</td>
</tr>
<tr>
<td>$T$ (aircraft)</td>
<td>(0.053), 0.110</td>
<td>2.15</td>
<td>5000</td>
<td>0.5</td>
</tr>
<tr>
<td>$U$ (1000)</td>
<td>0.105, 0.121, 0.114</td>
<td>1.93, 1.68, 1.80</td>
<td>12900, 11000, 12300</td>
<td>0.33, 0.32, 0.54</td>
</tr>
<tr>
<td>$U$ (700)</td>
<td>0.096, 0.104, 0.082</td>
<td>1.93, 1.86, 1.87</td>
<td>12000, 11000, 9000</td>
<td>0.24, 0.29, 0.83</td>
</tr>
<tr>
<td>$U$ (aircraft)</td>
<td>(0.046), 0.095</td>
<td>2.10</td>
<td>25000</td>
<td>0.8</td>
</tr>
<tr>
<td>$h$ (1000)</td>
<td>0.121, 0.109, 0.128</td>
<td>2.10, 1.81, 1.86</td>
<td>19800, 15900, 21700</td>
<td>0.33, 0.51, 0.46</td>
</tr>
<tr>
<td>$h$ (700)</td>
<td>0.094, 0.100, 0.091</td>
<td>1.75, 1.60, 1.74</td>
<td>11000, 11800, 9000</td>
<td>0.26, 0.37, 0.46</td>
</tr>
<tr>
<td>$h$ (aircraft)</td>
<td>(0.055), 0.114</td>
<td>2.10</td>
<td>10000</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 1b: An intercomparison of the 1000 mb fields, the triplets representing the parameter estimates for integrations of $t = 0, 48, 144$ hours.

<table>
<thead>
<tr>
<th>Field</th>
<th>$C_1$</th>
<th>$\alpha$</th>
<th>$L_{\text{eff}}$ (km)</th>
<th>$\delta$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$ (GEM)</td>
<td>0.125, 0.115, 0.112</td>
<td>1.64, 1.68, 1.69</td>
<td>25700, 20500, 25700</td>
<td>0.27, 0.26, 0.80</td>
</tr>
<tr>
<td>$T$ (GFS)</td>
<td>0.142, 0.138</td>
<td>1.72, 1.71</td>
<td>27900, 26000</td>
<td>0.59, 0.60</td>
</tr>
<tr>
<td>$U$ (GEM)</td>
<td>0.121, 0.122, 0.123</td>
<td>1.68, 1.62, 1.61</td>
<td>11000, 11000, 12300</td>
<td>0.32, 0.36, 1.24</td>
</tr>
<tr>
<td>$U$ (GFS)</td>
<td>0.114, 0.107</td>
<td>1.80, 1.84</td>
<td>12300, 11200</td>
<td>0.54, 0.64</td>
</tr>
<tr>
<td>$h$ (GEM)</td>
<td>0.109, 0.106, 0.112</td>
<td>1.81, 1.80, 1.77</td>
<td>15900, 13800, 14100</td>
<td>0.51, 0.49, 1.51</td>
</tr>
<tr>
<td>$h$ (GFS)</td>
<td>0.128, 0.128</td>
<td>1.86, 1.81</td>
<td>21700, 20900</td>
<td>0.46, 0.46</td>
</tr>
</tbody>
</table>

2.3 The Cascade structure of radiances:

2.3.1 Ground a space radar measurements of rain:

An early empirical test of multiplicative cascade models was made on radar reflectivities of rain (fig. 6 a, (Schertzer and Lovejoy, 1987)). This analysis extended over the range 1-128 km which is roughly the widest range possible for a single ground based radar. From the linearity of log $M_q$ versus log $\lambda$ shown in the figure, we see that it gives strong support for the multiplicative cascade idea. The radar reflectivity is proportional to the sum of the squares of the drop volumes so that it is nontrivially related to the rain rate (which is proportional to the sum of the products of the volumes with the vertical fall speeds). However – at least above a minimum detectable threshold
- the radar reflectivity is an accurately measured atmospheric signal and is strongly coupled with the rain rate so that the cascade structure of the reflectivities is strong evidence in favour of the cascade hypothesis for rain and the other fields.

In figure 6 a we see that the regressions over the observed scales apparently cross at about $\log_{10} \lambda = -0.2$ corresponding to a scale $32,000 \text{ km}$; from eq. 4 we see that this is expected at $\lambda = \lambda_{\text{eff}}$. This “effective outer scale” is the scale at which a pure multiplicative cascade would have to start in order to explain the observed variability over the measured range. The fact that it is of the order of the size of the atmosphere (the largest great circle distance is 20,000 $\text{km}$) is as expected; the fact that it is a little larger simply indicates that even at 20,000 $\text{km}$ there is a residual variability due to the interaction of rain with other fields. Although networks of ground based radar can be used to obtain continental scale reflectivities, the resulting mosaics involve large numbers of partially overlapping radars giving coverage which is far from uniform. In order to directly verify the cascade behaviour up to planetary scales, it is therefore more convenient to use satellite data. Fig. 6 b shows the result using the first orbiting weather radar, the PR instrument on the Tropical Rainfall Measuring and Mission (TRMM) satellite (Lovejoy et al., 2008a). From the figure, we see that again, the scaling (log-log linearity) is excellent, the main exception being for the low $q$ values. Adopting the convention that any number $x^0=1$ if $x \neq 0$ and $x^0=0$ if $x = 0$, we find that the $q = 0$ curve corresponds to the scaling of the raining areas. However the PR instrument has a very high minimum detectable signal: it is double the mean value and such thresholding breaks the scaling. In (Lovejoy et al., 2008a) with the help of numerical multifractal simulations this scale breaking (curved lines for low $q$) was simply reproduced as a simple threshold effect.

Fig. 6 a: The moments $M_q$ of the normalized radar reflectivity for 70 constant altitude radar maps at 3 $\text{km}$ altitude from the McGill weather radar (10 cm wavelength 1 $\text{km}$ pulse length). The basic figure was adapted from (Schertzer and Lovejoy, 1987) in
who added the straight lines converging to an outer scale at 32,000 km.

Fig. 6 b: The same as fig. 6 a except for the TRMM reflectivities (4.3 km resolution). The moments are for $q = 0, 0.1, 0.2, \ldots 2$, from (Lovejoy et al., 2009a). The poor scaling for the low $q$ values can be explained as artifacts of the fairly high minimum detectable signal. $L_{ref} = 20,000$ km so that $\lambda = 1$ corresponds to 20,000 km, the lines cross at the effective outer scale $\approx 32,000$ km.

2.3.2 Long and short wave radiances, passive microwaves: the scaling of the earth’s energy budget:

In addition to the PR instrument, the TRMM satellite had a visible and infrared (VIRS; 5 wavelengths) instrument as well as a passive microwave (TMI; 5 wavelengths) instrument. These were analyzed in (Lovejoy et al., 2009a), the corresponding analyses for the key energy containing short wave (visible) and long wave (thermal IR) wavelengths are shown in fig. 7 a, b. We see once again excellent scaling to within about the same degree of accuracy but with somewhat smaller outer scales; table 2 a, b shows the details and comparison with a much more limited earlier study ((Lovejoy, 2001)). These results are bolstered by those from thermal infrared data from the geostationary satellite MTSAT (fig 7 c, (Pinel, 2009)). Sections from 30°S to 40°N, about 13,000 km in the east-west over the Pacific ocean were used; every hour for two months (1440 images in all). It is interesting to note that the MTSAT analyses were
carried out in both east-west and north–south directions, fig. 7 c is the geometric mean, presumably closer to the TRMM analyses which were made along the satellite track, typically oriented N-E or S-W. In fig. 17 a we show the E-W MTSAT analysis (compared with the temporal analysis of the save data) showing that the scaling is not so good at larger scales – but that the NS statistics almost exactly compensate leading to the excellent scaling in fig. 7 c. The scaling behaviour of these radiances is consistent with the large scale cascade structure of the wind and temperature field. This is because it shows that while the corresponding ranges are clearly not the source and sink free regime postulated for a turbulent inertial range, it implies that the energy sources and sinks are themselves scaling so that the basic assumptions of the cascade model are still apparently satisfied. In addition, the radiances and cloud fields are strongly nonlinearly coupled so that the scaling of the radiances is in itself strong evidence for the scaling of the clouds and hence presumably the dynamics. In this regard the statistical physics problem of the interactions of radiation and scaling cloud fields is pertinent see e.g. (Lovejoy et al., 1990), (Davis et al., 1993), (Borde and Isaka, 1996), (Naud et al., 1996), (Schertzer et al., 1997), (Watson et al., 2008).

<table>
<thead>
<tr>
<th>Channel</th>
<th>Wavelength (µm)</th>
<th>Resolution (km)</th>
<th>δ(%)_line²</th>
<th>δ(%)_unit³</th>
<th>α</th>
<th>C₁</th>
<th>H</th>
<th>L_eff (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>VIRS 1</td>
<td>0.630</td>
<td>8.8</td>
<td>0.60</td>
<td>0.71</td>
<td>1.35</td>
<td>0.077</td>
<td>0.19</td>
<td>9800</td>
</tr>
<tr>
<td>VIRS 2</td>
<td>1.60</td>
<td>8.8</td>
<td>0.83</td>
<td>1.37</td>
<td>1.41</td>
<td>0.079</td>
<td>0.21</td>
<td>5000</td>
</tr>
<tr>
<td>VIRS 3</td>
<td>3.75</td>
<td>22</td>
<td>1.10</td>
<td>1.58</td>
<td>1.99</td>
<td>0.065</td>
<td>0.27</td>
<td>17800</td>
</tr>
<tr>
<td>VIRS 4</td>
<td>10.8</td>
<td>8.8</td>
<td>0.48</td>
<td>0.53</td>
<td>1.56</td>
<td>0.081</td>
<td>0.26</td>
<td>12600</td>
</tr>
<tr>
<td>VIRS 5</td>
<td>12.0</td>
<td>8.8</td>
<td>0.47</td>
<td>0.81</td>
<td>1.63</td>
<td>0.084</td>
<td>0.33</td>
<td>15800</td>
</tr>
<tr>
<td>AVHRR</td>
<td>0.58-</td>
<td>2.2</td>
<td></td>
<td></td>
<td>1.92</td>
<td>0.075</td>
<td>0.32</td>
<td>18700</td>
</tr>
<tr>
<td>14 vis</td>
<td>0.68</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>11.5-12.5</td>
<td>2.2</td>
<td></td>
<td></td>
<td>1.91</td>
<td>0.079</td>
<td>0.36</td>
<td>25200</td>
</tr>
<tr>
<td>AVHRR</td>
<td>14 IR</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2a: This shows the characteristics of the five visible and infra red channels (VIRS instrument, TRMM and two AVHRR channels). The H estimates are based on structure functions. The mean residues (δ, eq. 12) are given both with respect to the restrictive hypothesis that the cascades are universal multifractals (i.e. they respect the cascade eq. 4 with the universal form for K(q), eq. 2), and for the less restrictive hypothesis, that they only respect eq. 4.

1 These were from 153 visible, 214 IR scenes each 280X280 km over Oklahoma, from (Lovejoy, 2001) (Lovejoy and Schertzer, 2006).
2 This is the residual with respect to pure power law scaling.
3 This is the residual with respect to universal multifractal scaling with α = 1.5, C₁=0.08, only the outer scale is fit to each channel.
<table>
<thead>
<tr>
<th>Channel</th>
<th>Wavelength</th>
<th>Resolution (km)</th>
<th>$\Delta$(%) line$^2$</th>
<th>$\Delta$(%) uni$^3$</th>
<th>$\alpha$</th>
<th>$C_i$</th>
<th>$H$</th>
<th>$L_{eff}$ (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TMI 1</td>
<td>3.0 cm (10.6GHz)</td>
<td>111.4</td>
<td>1.40</td>
<td>1.55</td>
<td>1.35</td>
<td>0.255</td>
<td>0.50</td>
<td>15900</td>
</tr>
<tr>
<td>TMI 3</td>
<td>1.58 cm (19.35GHz)</td>
<td>55.6</td>
<td>1.71</td>
<td>1.93</td>
<td>1.76</td>
<td>0.193</td>
<td>0.331</td>
<td>6900.</td>
</tr>
<tr>
<td>TMI 5</td>
<td>1.43 cm (22.24GHz)</td>
<td>27.8</td>
<td>1.62</td>
<td>1.82</td>
<td>1.93</td>
<td>0.157</td>
<td>0.453</td>
<td>5000.</td>
</tr>
<tr>
<td>TMI 6</td>
<td>8.1 mm (37 GHz)</td>
<td>27.8</td>
<td>1.73</td>
<td>1.95</td>
<td>1.76</td>
<td>0.15</td>
<td>0.377</td>
<td>4400.</td>
</tr>
<tr>
<td>TMI 8</td>
<td>3.51 mm (85.5GHz)</td>
<td>13.9</td>
<td>1.40</td>
<td>1.70</td>
<td>1.90</td>
<td>0.102</td>
<td>0.238</td>
<td>6300.</td>
</tr>
<tr>
<td>TRMM Z</td>
<td>2.2 cm (13.2GHz)</td>
<td>4.3</td>
<td>6.0*</td>
<td>4.6*</td>
<td>1.50</td>
<td>0.63</td>
<td>0.00</td>
<td>32000</td>
</tr>
</tbody>
</table>

Table 2b: This shows the characteristics of the five TMI channels and the PR reflectivity from (Lovejoy et al., 2009a). All used vertical polarization. The $H$ estimates are based on structure functions.

Fig. 7 a: TRMM visible data (0.63 $\mu$m) from the VIRS instrument, channel 1 with fluxes estimated at 8.8 km. Only the well – lit 15,000 km orbit sections were used. $L_{ref} = 20,000$ km so that $\lambda = 1$ corresponds to 20,000 km, the lines cross at the effective outer scale $\approx 9,800$ km (from Lovejoy et al., 2009a)).
Fig. 7 b: Same as fig. 7 a except for VIRS thermal IR (channel 5, 12.0 µm),
external scale 15,800 km (see table 2 a for details, from (Lovejoy et al., 2009a)).
Fig. 7c: Logs of normalized moments $M_q$ versus $\log_{10} \lambda$ for 2 months (1440 images) of MTSAT, thermal IR, 30 km resolution over the region 40° N to 30° S, 130° east-west over the western Pacific, the geometric average of east-west and north-south analyses. $L_{ref} = 20,000$ km so that $\lambda = 1$ corresponds to 20,000 km, the lines cross at the effective outer scale ≈ 35,000 km (from (Pinel, 2009)).

2.4 Atmospheric boundary conditions: the topography

Physically, the PR signal comes purely from the atmosphere whereas the visible and infra red radiances depend on the states of both the atmosphere and surface. Just as various surface features affect the radiances, so they also affect the atmosphere; they are important boundary conditions for the atmosphere. If the topography had a strong characteristic scale then it could impose this on the atmospheric fields and break the scaling. Fig. 8a shows the spectral analysis of the largest statistical study of the topography to date showing that it has accurate spectral scaling (roughly $E(k) \approx k^{-2.1}$) over a range of roughly $10^5$ in scale. This is the latest update of the original $\approx k^2$ spectrum first proposed by (Venig-Meinesz, 1951), itself updated repeatedly over the last 50 years ((Balmino et al., 1973), (Bell, 1975), (Fox and Hayes, 1985), (Gibert and Courtillot, 1987), (Balmino, 1993)). Note that a pure multiplicative cascade has a spectrum $E(k) \approx k^\beta$ with $\beta = 1 - K(2)$; this is because spectra are fourier transforms of the autocorrelation function which is a second order ($q = 2$) moment. For observables related to the fluxes by eq. 3 a the extra $\Delta x^H$ corresponds to $k^H$ filtering so that $\beta = 1 - K(2)+2 H$.

Fig. 8 b shows the cascade structure of the topographic gradients obtained by combining four different data sets spanning the range 20,000 km down to sub-metric
scales. The cascade structure holds quite well until around 40 m. (Gagnon et al., 2006) argues that this break is due to the presence of trees (for the high resolution data set used over Germany, 40 m is roughly the horizontal scale at which typical vertical fluctuations in the topography are of the order of the height of a tree). Over the range of planetary scales down to ≈ 40 m, it was estimated that the mean residue of the universal scaling form (for all moments q ≤ 2) with parameters C₁ = 0.12, α = 1.79 was ±45% over this range of nearly 10⁵ in scale.

Fig. 8 a: A log-log plot of the spectral power as a function of wavenumber for four Digital Elevation Models. From right to left Lower Saxony (with trees, top), without trees (bottom), US (in grey), GTOPO30 and ETOP05. A reference line of slope -2.10 is shown for comparison. The small arrows show the frequency at which the spectra are not well estimated due to their limited dynamical range (for this and scale dependent corrections, see the original, (Gagnon et al., 2006)).
Fig. 8b: Log/log plot of the normalized trace moments $M_q$ versus the scale ratio $\lambda = L_{ref}/l$ (with $L_{ref} = 20000\ km$) for the three DEMs (circles correspond to ETOPO5, X’s to U.S. and squares to Lower Saxony). The solid lines are there to distinguish between each value of $q$ (from top to bottom, $q = 2.18, 1.77, 1.44, 1.17, 0.04, 0.12$ and $0.51$). The trace moments of the Lower Saxony DEM with trees for $q = 1.77$ and $q = 2.18$ are on the graph (indicated by arrows). The theoretical lines are computed with the global $K(q)$ function. Figure reproduced from (Gagnon et al., 2006).
2.5 Aircraft measurements of wind, temperature, humidity, pressure and potential temperature:

2.5.1 The biases in the wind statistics:

We mentioned that aircraft do not fly on perfectly flat trajectories, that over significant ranges of scale, their trajectories are typically fractal; this opens the possibility that their vertical fluctuations might significantly influence their measurements. This has been confirmed on both stratospheric flights at roughly constant Mach number (Lovejoy et al., 2004) and in tropospheric flights at roughly constant pressure levels (Lovejoy et al., 2009d). The latter results are particularly pertinent since all the published tropospheric turbulence campaigns have used data from roughly isobaric flights and because it gives a detailed analysis based on 24 flight segments (“legs”) of the Gulfstream 4 (NOAA) aircraft each 1120 km long at 1s (≈ 280 m) resolution (this experiment is described in detail in (Hovde et al., 2009); it involved 10 aircraft flights over a roughly 2 week period over the northern Pacific each dropping the 20 - 30 drop sondes.

Fig. 9 shows the ensemble spectra for the altitude, pressure, longitudinal and transverse wind components and from the temperature and humidity. For clarity, the spectra are displaced in the vertical and in order to amplify the deviations from $k^{-5/3}$ scaling they have been normalized or “compensated” by dividing by the theoretical Kolmogorov spectrum ($k^{-5/3}$). Flat regions thus have spectra $\approx k^{-5/3}$. In addition, in order to show the behaviour more clearly - with the exception of the lowest 10 wavenumbers - we have averaged the spectrum over logarithmically spaced bins, 10 per order of magnitude. From the figure we can make out the following features a) the altitude and pressure spectra show that there are three regimes characterizing the trajectory (roughly < 3 km, 3-40 km and >40 km), b) there are two regimes characterizing the wind (roughly < 40 km and >40 km) and c) a single regime for temperature and humidity. Detailed leg by leg analysis shows that the exact transition point to the large scale regime varies considerably from leg to leg, 40 km being only an average. In order to understand this better, (Lovejoy et al., 2009d) considered the spectral coherencies and phase relations between the altitude and pressure wind, temperature and humidity. We summarize their overall conclusions as functions of scale range:
Fig. 9: Top top bottom: this shows the compensated pressure (red), altitude (green), east-west, north south winds (middle), humidity and temperature (blue, orange, bottom). Reference slopes corresponds to $k^{-5/3}$ (flat), $k^{-2.4}$ and $k^{-2}$. The spectra are for 24 legs each 1120 km long, averaged over 10 per order of magnitude. Units of $k$: (km)$^{-1}$. Adapted from (Lovejoy et al., 2009d).

i) $k > (3 \text{ km})^{-1}$:

At these small scales, due to the inertia of the aircraft (which prevents it from rapidly responding to changes in wind), the coherency and phase are not statistically significant with respect to the altitude. At these small scales the trajectory is particularly smooth; the corresponding section of the altitude spectrum in fig. 9 is particularly steep. However, the situation is more interesting for the pressure; the coherency/phase analysis shows that the transverse component is significantly coherent with respect to the pressure, and that the phase of the pressure lagging behind the wind fluctuations. This is presumably the effect of fluctuations in the “dynamical pressure” ($1/2 \rho v^2$) caused by the wind changes.

ii) $(40 \text{ km})^{-1} < k < (3 \text{ km})^{-1}$:

Moving to lower wavenumbers $k < (3 \text{ km})^{-1}$; we see from the figure that the altitude and pressure are almost exactly Kolmogorov ($k^{-5/3}$); this is the fractal trajectory region. Coherency and phase analyses show that for the longitudinal component, there are very strong coherencies and phase relations for essentially all the scales larger than 3 km with the relation between pressure and wind a bit stronger than that between altitude and wind, especially the longitudinal component. When we consider the phase relations, we find that whereas the pressure continues to lag behind the wind, the wind lags behind
the altitude changes. This could be a consequence of the autopilot (on a time scale of 10 -100 s) adjusting the level due to the smaller scale turbulent trajectory fluctuations.

iii) $k < (40 \text{ km})^{-1}-(60 \text{ km})^{-1}$:

At the larger scales, the pressure and the altitude no longer follow $k^{-5/3}$ spectra (fig. 9). In this region, the coherency/phase analysis shows that the phases of both the altitude and pressure with respect to the longitudinal component reverse sign. In this regime, the pressure leads the wind fluctuations while the altitude lags behind. This is presumably the regime in which the aircraft most closely follows the isobars. From fig. 9, we see that this is also the regime where the wind spectrum follows the $k^{-2.4}$ rather than $k^{-5/3}$ law; (Lovejoy et al., 2009d) argue that it is this “imposed” vertical displacement that leads to the spurious appearance of the vertical spectral exponent $\approx 2.4$ (see the discussion of the vertical statistics in section 3). It is significant that detailed re-examination of all the major tropospheric campaigns (GASP, (Gage and Nastrom, 1986), MOZAIC, (Cho and Lindborg, 2001), (Lindborg and Cho, 2001) and also (Gao and Meriwether, 1998)) display nearly identical statistics i.e. transitions from $\approx k^{-5/3}$ to $\approx k^{-2.4}$ behaviours at average scales 30 - 50 km. Finally, application of the anisotropic scaling model on the individual flight legs discussed in section 3 shows that it can explain the first order structure function statistics to within $\pm 7\%$ over the range 0.28 - 500 km so that the break is indeed likely spurious.

It is interesting that the temperature and coherency/phase analysis shows that over the regime $(40 \text{ km})^{-1}<k<(3 \text{ km})^{-1}$ there are only low coherencies and small phases for both, becoming statistically insignificant for $k<(100 \text{ km})^{-1}$. The most significant - the temperature phases - indicate that there is a lag with respect to the altitude, as expected if the altitude fluctuations were imposed. The overall weak link between the trajectory statistics and the temperature and humidity fluctuations is thus consistent with their excellent spectral scaling $k^{-\beta}$ (with $\beta = 2.13, 2.10$ respectively) over the entire range (fig. 9).

2.5.2 Aircraft estimates of horizontal cascade parameters:

From the coherency and phase analyses it at least plausible that the aircraft can adequately determine horizontal temperature, humidity and potential temperature statistics – since these have low coherencies with the altitude and pressure – but also the transverse wind – at least over the range where the coherence is low, i.e. up to about 30 – 50 km. Beyond that, we expect that the aircraft will spuriously measure the vertical wind statistics; e.g. the spectrum $k^{-2.4}$ indicated in fig. 9.

We can now refer the reader to fig. 10 which shows the flux analysis results for the longitudinal, transverse wind, altitude, pressure, temperature and humidity, potential temperature. Starting with the results for the altitude (fig. 10 c), we see that its variability makes a sharp transition at around 30 - 50 kilometers: the small scales being extremely intermittent, the larger scales much less so. Similarly, although it is less marked, the pressure (fig. 10 d) also shows a transition in its variability at the same scale. This gives further insight into the observed transition in the wind spectra at this scale (fig. 9). Considering the other fields, we see fairly convincing cascade structure for the wind (fig. 10 a, b) – at least up to the 30 - 50 km scale while (as in fig. 9) the temperature, humidity and potential temperature (fig. 10 e, f, g) show excellent scaling throughout. We could also mention that the outer scales (except for altitude) are of the
order of the size of the earth although the wind is somewhat larger – the variability being presumably increased by the variability of the altitude. Table 3 compares the parameter estimates.
Fig. 10 a, b, c, d, e, f: These show the normalized moments for longitudinal (top left a), transverse (top right, b), z (second row left, c, note different scale), pressure (middle right, d), temperature (third row left, e), humidity, third row right, f.

Fig. 10g: Same but log potential temperature. From (Lovejoy et al., 2009b).

<table>
<thead>
<tr>
<th></th>
<th>T</th>
<th>Logθ</th>
<th>h</th>
<th>v_{long}</th>
<th>v_{trans}</th>
<th>p</th>
<th>z</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>0.60±0.04</td>
<td>0.59±0.03</td>
<td>0.61±0.007</td>
<td>0.46</td>
<td>0.37</td>
<td>0.36</td>
<td>0.43</td>
<td>0.33</td>
</tr>
<tr>
<td>C_1</td>
<td>0.053±0.006</td>
<td>0.048±0.005</td>
<td>0.055±0.001</td>
<td>0.033</td>
<td>0.046</td>
<td>0.031</td>
<td>0.068</td>
<td>0.076</td>
</tr>
<tr>
<td>α</td>
<td>2.15</td>
<td>2.20</td>
<td>2.10</td>
<td>2.10</td>
<td>2.10</td>
<td>2.2</td>
<td>2.15</td>
<td>1.83</td>
</tr>
<tr>
<td>L_{eff}(km)</td>
<td>50000</td>
<td>10000</td>
<td>10000</td>
<td>10^5</td>
<td>25000</td>
<td>1600</td>
<td>50</td>
<td>25000</td>
</tr>
<tr>
<td>δ</td>
<td>0.5</td>
<td>2.0</td>
<td>0.5</td>
<td>0.4</td>
<td>0.8</td>
<td>0.5</td>
<td>2.6</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 3: Horizontal parameter estimates. These are over the range 100 km down to 2 km except for $z$ which is over the range 20 km to 0.5 km. Error estimates only for those which are apparently unaffected by aircraft trajectory, they are half the difference of parameters when estimated over the range 200 km to 20 km and 20 km to 2 km. Note that the aircraft $\alpha$ estimates are a bit too big since the theoretical maximum is $\alpha = 2$. They were estimated with the double trance moment technique which depends largely on the statistics of the weaker events and these could be affected by aircraft turbulence. The $H$ parameters were estimated from spectral exponent $\beta$ and the value $K(2)$ using the equation $H = (\beta-1+K(2))/2$. Since the humidity is very low at the aircraft altitude, the equivalent potential temperature was extremely close to the potential temperature, hence the statistics were indistinguishable and are therefore not explicitly given in the table.
3. **The vertical stratification**

3.1 Discussion:

The horizontal cascade structures discussed in section 2 covered scales starting near those of the planet, hence if they are realizations of an isotropic turbulent process, it must be two dimensional. However, the same scaling regimes continue on down to scales much smaller than the scale thickness ($\approx 10 \text{ km}$) - in the case of aircraft and lidar - down to $\approx 100 - 300 \text{ m}$ which is much too small to be part of a 2-D turbulent regime. We are therefore lead to the conclusion that atmospheric scaling cannot have the same exponents in the vertical as in the horizontal.

Let us consider a fairly general case of anisotropic but scaling turbulence so that the fluctuations in the horizontal velocity over a horizontal lag $\Delta x$ and vertical lag $\Delta z$ follow:

$$
\Delta v = \phi_h \Delta x^{H_h}; \quad \Delta v = \phi_v \Delta z^{H_v}
$$

where $\phi_h$, $\phi_v$ are the turbulent fluxes dominant in the horizontal and vertical directions respectively and $H_h$, $H_v$ are the corresponding exponents. The (isotropic) Kolmogorov law is recovered with $\phi_h = \phi_v = \varepsilon^{1/3}$, $H_h = H_v = 1/3$ where $\varepsilon$ is the energy flux. In comparison, the $23/9$ D model of anisotropic turbulence (Schertzer and Lovejoy, 1985b) in which the horizontal is dominated by the energy flux ($\varepsilon$, $m^2 s^{-3}$) and the vertical by buoyancy variance flux ($\phi$, $m^2 s^{-5}$) is obtained with $\phi_h = \varepsilon^{1/3}$, $\phi_v = \phi^{1/5}$, $H_h = 1/3$, $H_v = 3/5$. Similarly, the popular quasi-linear gravity wave models (Dewan, 1997; Dewan and Good, 1986), (Gardner, 1994; Gardner et al., 1993) typically take $\phi_h = \varepsilon^{1/3}$, $\phi_v = N$ so that $H_h = 1/3$, $H_v = 1$ (the Brunt Väisälä frequency; actually this is not a turbulent flux, a fact which is a serious weakness of that theory).

In order to write these anisotropic models in a form valid for any vector $\Delta r = (\Delta x, \Delta y, \Delta z)$ we can use the formalism of Generalized Scale Invariance (Schertzer and Lovejoy, 1985a) and write:

$$
\Delta v = \phi_s (\|\Delta r\|)^{\phi_s}
$$

where the scale function (indicated by the double bars) replaces the usual vector norm appropriate for isotropic turbulence. In general, the scale function satisfies the scale equation:

$$
\|T_\lambda \Delta r\| = \lambda^{-1} \|\Delta r\|; \quad T_\lambda = \lambda^{-G_s}
$$

where $T_\lambda$ is the scale changing operator and $G_s$ is its spatial generator. For the case of pure stratification in the vertical plane and assume horizontal isotropy, we take:

$$
G_s = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & H_z
\end{pmatrix}; \quad H_z = \frac{H_h}{H_v}
$$

(see section 4 for the generalization to space-time). When $G_s$ is a matrix (corresponding to linear group generators), the notion of scale is position independent. When the generator is nonlinear, the $G$'s are more general nonlinear operators which will depend on the coordinates; the notion of scale will be position dependent. For the case in which
$G$ is the identity matrix, we have the usual isotropic, self-similar scale changes. In the case of “linear GSI”, where $G$ is a diagonal matrix, the system is “self affine” and we obtain stratification along a coordinate axes (e.g. fig. 3 b and 11). Finally, when $G$ has off-diagonal elements (fig. 3 c) we have differential rotation and stratification. The idea is that the basic dynamical symmetries determine the $G$’s and the scale function is then determined by solving the functional scale equation 15 for specific boundary conditions, i.e. by specifying all the unit vectors (the “unit balls”; see below). By acting on the unit vectors in this way $T_\lambda$ then generates all the other vectors, it determines the scale.

Figures 3 and 11 are visual illustrations of the “phenomenological fallacy” (Lovejoy and Schertzer, 2007) i.e. the confounding of mechanism and morphology. This is because GSI allows for quite striking morphologies which evolve with scale but yet are all generated by the same basic scale invariant dynamic (via $G$). GSI concretely demonstrates that one cannot make this identification.

We may now define $D_s = \text{Trace } G_s$ and $D_{st} = \text{Trace } G_{st} = D_s + H_z$ as the “elliptical dimensions” characterizing the spatial and space-time anisotropies respectively. With the above dimensionally determined exponents we find $D_s = 23/9$, $D_{st} = 29/9$. The $D$’s are dimensions since changing the scales of the vectors by $\lambda$ (by operating with $\lambda^{-G}$) changes their volumes by $\det(\lambda^{-G}) = \lambda^{-\text{Trace}(G)}$, they are therefore exponents which quantify the change of volume with scale.

In the vertical ($x, z$) plane, a simple (“canonical”) solution of the scale equation 15 is:

$$
\|\Delta r\| = l_s \left( \left( \frac{\Delta x}{l_x} \right)^2 + \left( \frac{\Delta z}{l_z} \right)^2 \right)^{H_z/2} ; \quad H_z = \frac{H_h}{H_v} ; \quad l_s = \left( \frac{\varphi_h}{\varphi_v} \right)^{1/(H_v-H_h)}
$$

where $H_z$ is the exponent characterizing the degree of stratification ($H_z = 1$ corresponds to isotropic 3D turbulence, $H_z = 0$ to isotropic 2-D turbulence) and $l_s$ is the “sphero-scale” so-called because the structures are roundish at that scale; it is the unique scale defined by the fluxes $\varphi_h$, $\varphi_v$. The scale function need only satisfy the fairly general scale equation, so that the above “canonical” form is only the simplest scale function but is adequate for our purposes. It can be verified that if we successively take $\Delta r = (\Delta x, 0)$ and $\Delta r = (0, \Delta z)$ that we recover eq. 13.

In this framework the quasi-linear gravity wave model has $H_z = 1/3$ and therefore $D_{el} = 7/3$ and we have noted that the classical 2D and 3D isotropic turbulences have $H_z = 0, 1$ hence $D_{el} = 2, 3$ respectively. The 23/9 $D$ model of stratification was found to be obeyed quite precisely for passive scalar densities estimated by lidar (i.e. with the above scale function replacing the vector norm in the isotropic Corssin-Obukhov law of passive scalar advection ((Lilley et al., 2004), (Lilley et al., 2008); c.f. $H_z = 0.55\pm0.02$). If $0 < H_z < 1$, then structures larger than the sphero-scale become progressively flatter at larger and larger scales; see the simulations in fig. 11.
Fig. 11: A sequence “zooming” into vertical cross section of an anisotropic multifractal cloud with $H_z = 5/9$. Starting at the upper left corner, moving from left to right, from top to bottom, we progressively zoom in by factors of 1.21 (total factor $\approx 1000$). Notice that while at large scales, the clouds are strongly horizontally stratified, when viewed close up they show structures in the opposite direction. The sphero-scale is equal to the vertical scale in the left most simulation on the bottom row. The multifractal parameters were $H = 1/3$, $C_1 = 0.1$, $\alpha = 1.8$.

3.2 The implications for aircraft statistics

We may now revisit the aircraft measurements analyzed above. In order to understand the effect of the vertical trajectory variability on the horizontal wind statistics, consider a large scale section roughly following a sloping isobar with slope $s$: 
\[ \Delta v = \phi_h l_s^{H_h} \left( \frac{\Delta x}{l_s} \right)^2 + \left( \frac{s \Delta x}{l_s} \right)^{2/H_h} \]  

When considering the ER-2 trajectory, (Lovejoy et al., 2004) pointed out that if \( s \) was constant, then there would exist a critical lag \( \Delta x_c = l_s s^{1/(H_h-1)} \) such that for \( \Delta x > \Delta x_c \), the second term would dominate the first and we would obtain:

\[ \Delta v = \phi_h \Delta x^{H_h} ; \quad \Delta x \ll \Delta x_c \]
\[ \Delta v = \phi_v s^{H_v} \Delta x^{H_v} ; \quad \Delta x \gg \Delta x_c \]  

We would therefore expect a spurious break in the horizontal scaling at \( \Delta x_c \) after which the aircraft would measure the vertical rather than horizontal statistics with exponent \( H_v \) rather than \( H_h \). In the spectra, this transition corresponds to a transition from \( k^{-5/3} (H = 1/3) \) to \( k^{-2.4} (H \approx 0.7) \); fig. 9 shows that this is indeed a good estimate of the small wavenumber part of the spectrum, see the discussion in section 2.5.

3.3 Direct verification of the anisotropic cascades using lidar backscatter of aerosols:

3.3.1 Simultaneous horizontal and vertical analyses:

The difficulty with using aircraft data to understand the nature of atmospheric stratification is that even if we overcome the aircraft measurement problems, they must be compared to in situ data from a different instrument in the vertical (unless perhaps we use ascent and descent legs, c.f (Tuck et al., 2004)). Clearly, the ideal way to study the vertical stratification is through remotely sensed vertical sections, hence we start our investigation of the vertical cascade structure by exploiting a unique dataset of airborne lidar backscatter. The data were taken over three afternoons in Aug. 2002 near Vancouver, British Columbia (see (Radkevitch et al., 2007) for more information on the lidar), see fig. 12 a, and the zoom, 12 b for an example showing the extremely fine details, including hints that at the small scales that structures are no longer flat, but start to be stretched in the vertical (fig. 12 b, compare this with the simulation in figure 11). The lidar backscatter is primarily from aerosols; (Lilley et al., 2004) compared the first order horizontal and vertical structure functions, and (Radkevitch et al., 2007), (Radkevitch et al., 2008) studied the corresponding spectra, including a new anisotropic scaling analysis technique (ASAT) involving nonlinear coordinate transformations. (Lilley et al., 2008) contains a literature review and additional anisotropy analyses including of the fluxes estimated from gradients of the backscatter. The conclusions were broadly that the backscatter statistics can be accurately described if the ratio of horizontal to vertical scaling exponents was \( H_z \approx 0.55 \). In addition, the scale at which horizontal and vertical fluctuations are of equal magnitude (“the spherico-scale”) was directly estimated for the first time (it varied between about 10 and 80 cm). \( H_z \) was close to the value 5/9 discussed earlier.
Fig. 12 a: This is a vertical cross-section of lidar backscatter on 14 August 2001. The scale on the bottom is a logarithmic color scale: darker is for smaller backscatter (aerosol density surrogate), lighter is for larger backscatter. The vertical is 4.5 km and the horizontal is 120 km. The horizontal resolution is 100 m and the vertical resolution is 3 m. The range of scales in this data set is 1200 x 1500 (horizontal X vertical). The region in the red rectangle is blown up in fig. 12 b. The black shapes along the bottom are mountains in the British Columbia region.
Fig. 12 b: A blow up of the region within the red rectangle in fig. 12 a, it is 40 \textit{km} wide and 1000 \textit{m} thick. This panel highlights the high spatial resolution and the wide dynamic range. There is no saturated signal and high sensitivity to low signal return. Note that while at large scales, the structures are horizontally flat, at the smaller scales, we can begin to see structures that are more roundish, or even vertically aligned; compare with fig. 11. This and fig. 12 a are courtesy of K. Strawbridge. For other similar examples, see (Lilley et al., 2004).

A direct horizontal/vertical intercomparison of the normalized fluxes $M_q$ is given in fig. 13 a, b. We see that the cascade structure predicted by eq. 4 is well respected: not only are the lines quite straight, they also “point” to the effective outer scale of the process, - i.e. the scale at which a multiplicative cascade would have to start in order to account for the statistics over the observed range. We see that as before, for the horizontal analysis, $L_{\text{eff}}$ is a little larger than the physical scales ($\approx 25,000 \text{ km}, 50 \text{ km}$ for the horizontal and vertical respectively). Table 4 shows some of the parameters
characterizing $K(q)$ and shows that they are indeed quite different for the horizontal and vertical directions.

### 10 Vertical lidar cross-sections backscatter ratio.

<table>
<thead>
<tr>
<th>Field</th>
<th>Resolution (m)</th>
<th>$\bar{\delta}$ (%)</th>
<th>$C_1$</th>
<th>$H$</th>
<th>$\alpha$</th>
<th>$L_{eff}$ (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B vertical</td>
<td>12m x192m</td>
<td>0.4</td>
<td>0.11</td>
<td>0.60</td>
<td>1.82</td>
<td>50</td>
</tr>
<tr>
<td>B horizontal</td>
<td>12m x192m</td>
<td>0.5</td>
<td>0.076</td>
<td>0.33</td>
<td>1.83</td>
<td>25000</td>
</tr>
</tbody>
</table>

Table 4: This shows the cascade parameters. Table from (Lovejoy et al., 2009c).

---

Fig. 13 a: Horizontal analysis of the moments of the normalized lidar backscatter ratio for 10 atmospheric vertical cross-sections ($L_{ref} = 20,000$ km corresponding to $\lambda = 1$). The curves are for the moments of order $q = 0.2$, 0.4, …2. The largest directly accessible scale is $\approx 100$ km, and the lines converge to an effective outer scale of $L_{eff} \approx 25,000$ km. Reproduced from (Lovejoy et al., 2009c).
Fig. 13 b: The same cross-sections as in fig. 13 a but analyzed in the vertical direction, \( (L_{\text{ref}} = 10 \text{ km} \text{ corresponding to } \lambda = 1) \). The largest directly accessible scale is \( \approx 3 \text{ km} \), the point of convergence is \( L_{\text{eff}} \approx 50 \text{ km} \), see table 1. Note that the vertical axis is not the same as for the horizontal analysis; this is a consequence of the scaling anisotropy; the exponents are roughly in a constant ratio (reproduced from (Lovejoy et al., 2009c)).
Fig. 13 c: A space-space (horizontal/vertical) diagramme from 9 vertical lidar sections obtained from first order structure functions. The dashed lines have theoretical slopes 5/9, the thick black line is the bisectrix (x = y). The sphero-scale is the intersection of the empirical lines with the bisectrix. It can be seen that the sphero-scales are somewhat variable, but mostly between 10 cm and 1 m. At the larger scales, we see that the earth size (20,000 km) roughly corresponds to the troposphere thickness (10 km). Note that the empirical statistics are not so good at the largest scales. We thank A. Radkevitch for help with this analysis.

3.3.2 The construction of space-space diagrams from lidar data:

If we define a “structure” in a field f as a fluctuation in the value of f of magnitude \( \Delta f \), then we can use this to statistically define the relation between the horizontal and vertical extents of structures. For example, using the first order structure function we can equate the horizontal and vertical fluctuations:

\[
\langle |\Delta f(\Delta x)| \rangle = \langle |\Delta f(\Delta z(\Delta x))| \rangle
\]

which gives an implicit relation \( \Delta z(\Delta x) \) between the horizontal and vertical extents (\( \Delta x \), \( \Delta z \) respectively). Using the scale function (eq. 17) and the relation 14 for the fluctuation in terms of the scale function, we see that this is equivalent to using \( \| (\Delta x, 0) \| = \| (0, \Delta z(\Delta x)) \| \) or \( \Delta z = l_x (\Delta x / l_x)'' \). The same idea is used
in section 4 on space-time cross-section data to produce classical “space-time” (“Stommel”) diagrammes, so that here we use the expression “space-space” diagrammes. The existence of spatial vertical lidar cross section data spanning many orders of magnitude in scale allows us to empirically determine this statistical correspondence directly and accurately.

Fig. 13 c shows the result using 9 vertical cross-sections (using the same data sets that were described and analyzed in (Lilley et al., 2004)). We see that on a log-log plot the inferred log \( \Delta x \)-log \( \Delta z \) relationship is reasonably linear and that the slope is very near the theoretical value \( H_z = 5/9 \) (shown by reference lines; the scaling is not as good at the larger distances where the statistics are poor). However, what is particularly striking about the figure are the implications of extrapolating the lines both to larger and to smaller scales. First, at smaller scales, we see can estimate the sphero-scale (where \( \Delta x = \Delta z = l_s \)); we find it in the range 20 \( cm \) to 2 \( m \); similar to the other estimates discussed above \( l_s \) is determined by the intersection of the extrapolation of the empirical line with the solid black reference line, \( \Delta x = \Delta z \); it thus seems that the extrapolation is quite reasonable down to metric scales or less. However equally impressive is the extrapolation to larger scales: we see that extrapolation to the planetary scale (20,000 \( km \)) gives a corresponding vertical extent of \( \approx 10 \ km \) i.e. the thickness of the troposphere. In other words, there is no obvious reason why the scaling stratification should have a break anywhere in the meteorologically significant range of scales. Note that there are various ways to generalize and extend the method for estimating space-time relations. For example, we could use the same method to determine the horizontal/vertical relations for weak and strong events by considering structure functions with exponents \( q < 1 \) or \( q > 1 \) (see an example of this method in section 4). Alternatively, we could use the statistics of the fluxes (the normalized moments \( M_q \)) to establish the relation using \( M_q(\Delta x) = M_q(\Delta z(\Delta x)) \); this method is used extensively in section 4.

### 3.4 Vertical cascades using drop sondes:

#### 3.4.1 Description of the data set

The lidar data analysed in section 3.3 are uniformly spaced in orthogonal directions with high signal to noise ratios and are thus relatively straightforward to analyze. However if we seek to study the usual dynamic or thermodynamic variables, we are forced to turn to in situ measurements. Traditionally over a substantial part of the troposphere, radiosondes have been the only way to get such vertical information. However, they have numerous problems including payloads swinging into and out of the balloon’s wake, low vertical resolutions (typically of the order of 100 \( m \)) and slow ascent speeds which – in areas of strong downdrafts - can even temporarily become descents. As mentioned earlier, these technical difficulties have contributed to the absence of consensus on the nature of the vertical stratification.

In the last ten years, the development of GPS dropsondes has drastically changed this situation (Hock and Franklin, 1999). Dropsondes are free of problems with swinging payloads and wakes and they have rapid descent times (less than 15 minutes from the top of the troposphere) and - with the help of GPS tracking – they have high vertical resolutions (of the order of 5 \( m \), although see the discussion below). The data
discussed here were part of the same Winter Storms 2004 experimental discussed in section 2.5 using the NOAA Gulfstream 4 aircraft. During a 2-week period, ten flights each dropped 20-30 sondes, a total of 262. Of these, 237 reasonably complete sets were analyzed in (Lovejoy et al., 2007), (Lovejoy et al., 2008c), here we consider the 220 sonde subset which started at altitudes > 10 km and we summarize the findings of (Lovejoy et al., 2009c).

3.4.2 Intermittent multifractal sampling: The problem of outages

The cascade structure eq. 1 is the consequence of variability building up scale by scale over a potentially large scale ratio $\lambda$. In order to verify eq. 1 and to estimate $K(q)$, we attempt to invert the cascade process by systematically degrading the resolution of the fluxes by averaging. This is straightforward enough for data sampled at regular intervals, but for data with highly irregular resolutions we must take into account the variability of the resolution. The drop sonde resolution is variable for two reasons: first, even if the sampling was at the nominal 0.5 s, the variable vertical sonde fall speed would lead to variable vertical sampling intervals. This source of variability is not too large: the mean vertical sonde velocity decreases from about 18 m/s to about 9 m/s near the surface (due to increased air resistance); turbulence induced fluctuations increase this range of resolution by about another factor of 2. However, the variability problem is made much worse because of the outages even though these affected only 9.5% of the observations. The problem is that they affected every sonde, they were highly clustered and sometimes very large (i.e. occasionally several km in size).

By treating the measurement intervals in both the vertical and in time (Lovejoy et al., 2009c) obtained a surprising result: the outages had almost exact cascade structures with rather large intermittencies ($C_{time} \approx 0.21$, $C_{vert} \approx 0.23$) with outer scales near the outer scale of the data ($\approx 200$ s, $\approx 3$ km respectively). In order to overcome this outage problem, two developments are needed. The first is a robust technique to estimate the fluxes, the second a method of statistical correction to the scaling exponents in order to correct for the strongly variable resolutions ((Lovejoy et al., 2009c)).

When these techniques were applied to the drop-sonde data, the moments displayed in fig. 14 a, b were obtained. The quantities that we analysed can be roughly grouped into two categories: dynamical and thermodynamical variables. The dynamic variables (fig. 14 a) were the modulus of the horizontal wind $v$, the pressure $p$, the total air density ($\rho$, including that due to humidity), and the sonde vertical velocity $w_s$. We also separately analyzed the north - south and east - west components of the horizontal wind but the results were not much different and we will not discuss them further. For the vertical sonde velocity, the fluctuations around a quadratic fit (corresponding to a constant deceleration from 18 m/s to 9 m/s) were used. Due to the parachute drag, the fluctuations in $w_s$ depend on both the vertical and horizontal wind so that it should not be used as a surrogate for the vertical wind.

The thermodynamic variables are temperature ($T$), log potential temperature (log$\theta$), log equivalent potential temperature (log$\theta_E$) and humidity ($h$), see fig. 14 b. The log$\theta$ and log$\theta_E$ are proportional to the entropy densities of the dry and humid air respectively. In addition, their structure is important for the overall atmospheric (static) stability, for example $gd\log\theta/dz$ is the square of the Brunt – Väisälä frequency so that
where the latter is negative, the atmosphere is considered conditionally unstable. Similarly, it is convectively unstable when $g d \log \theta_E / d z$ is negative.

Fig. 14 a: The dynamical fields $v, p, \rho, w_s$ (clockwise from upper left) for $q = 0.2, 0.4, ..2$. Reproduced from (Lovejoy et al., 2009c).
From the figures we can see that with small deviations (c.f. the table 5) all the fields have small residuals with respect to the predictions of cascade theories (eq. 4; the residuals (eq. 12) were averaged over the range 1 km to 10 m).

<table>
<thead>
<tr>
<th></th>
<th>T</th>
<th>Logθ</th>
<th>LogθE</th>
<th>h</th>
<th>ν</th>
<th>p</th>
<th>ρ</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1.07±0.18</td>
<td>1.07±0.18</td>
<td>0.87±0.10</td>
<td>0.78±0.07</td>
<td>0.75±0.05</td>
<td>1.95±0.02</td>
<td>1.31±0.12</td>
</tr>
<tr>
<td>C₁</td>
<td>0.072</td>
<td>0.071</td>
<td>0.069</td>
<td>0.091</td>
<td>0.088</td>
<td>0.072</td>
<td>0.077</td>
</tr>
<tr>
<td>α</td>
<td>1.70</td>
<td>1.90</td>
<td>1.90</td>
<td>1.85</td>
<td>1.90</td>
<td>1.85</td>
<td>1.95</td>
</tr>
<tr>
<td>L_{eff} (km)</td>
<td>5.0</td>
<td>4.0</td>
<td>25.</td>
<td>16.</td>
<td>1.3</td>
<td>5.0</td>
<td>13</td>
</tr>
<tr>
<td>δ</td>
<td>1.4</td>
<td>1.2</td>
<td>1.9</td>
<td>1.4</td>
<td>2.3</td>
<td>1.1</td>
<td>1.4</td>
</tr>
</tbody>
</table>

Table 5: Vertical parameter estimates: The above are the vertical parameters from drop sondes, corrected for the sonde intermittency and for C₁, L_{eff} they are estimates for the part of the trajectory above 6 km. Due to the low humidity at 200 mb, the potential temperature is almost identical to the equivalent potential temperature and was not considered further.

The $H$ exponents are from the fluctuation analyses (see (Lovejoy et al., 2009c)) and are the means of the fits from 30 - 300 m and 300 – 3000 m, the spread is half the difference. The scale ratio $\lambda$ was defined with respect to 10 km. $\alpha$ was estimated to the
nearest 0.05 using the double trace moment technique which involves repeating the cascade analysis but with the flux raised to a series of different powers (Lavallée et al., 1993). All the regressions were taken over range $1.5 < \log_{10} \lambda < 3$ with $\lambda$ defined as the ratio of the reference scale 10 km to the resolution scale (i.e. corresponding to 300 m to 10 m).

### 3.5 Intercomparison of the stratification of different fields, estimating $H_z$, $D_{el}$:

We have shown that atmospheric fields are scaling over much of the meteorologically significant range in both the horizontal and vertical so that the dynamics are scaling, turbulent but anisotropic. The simplest anisotropic turbulence model involves a unique scale function for all the fields. This would imply that the ratio of horizontal and vertical components is $H_{hor}/H_{ver} = H_z = \text{constant}$, so that for universal multifractals $\alpha_{hor} = \alpha_{ver}$ and $C_{1hor}/C_{1ver} = H_z$. In (Lilley et al., 2008) there was an extensive analysis of this for the lidar data reviewed in section 3.3.

Combining the results from the aircraft and the drop sonde and taking into account a small apparent altitude dependence of the sonde exponents (so as estimate them at the 200 mb aircraft level), we obtain table 6. It should be noted that although in table 6 we give the ratio of the $C_1$ values, that since their values are small, their relative errors are large and consequently their ratios have large uncertainties. Since the $H$’s are larger, the ratio $H_{hor}/H_{ver}$ is more reliable than $C_{1hor}/C_{1ver}$, indeed in the latter case the error is very hard to reliably estimate and is not indicated except in the lidar case. The main conclusion of is that in the lower troposphere is heavily dependent on the humidity - that $T$, log $\theta$ and $B$ are within a standard error bar of the 23/9D result $H_z = 5/9$ whereas $h$ and the strongly humidity - dependent log $\theta_E$ have values around 0.7. At the same time, the velocity field is apparently anomalously low; this was the conclusion of detailed trajectory by trajectory analysis of aircraft data in (Lovejoy et al., 2009d), see the summary section 2.5.

If the ratios in table 6 are taken at face value then we are lead to the conclusion that two or more scale functions are required to specify the scale of atmospheric structures. While this is certainly possible, let us for the moment underline the various difficulties in obtaining the in situ estimates: the nontrivial vertical outages, the nontrivial aircraft trajectory fluctuations. In addition, detailed analysis of the altitude dependence of the horizontal velocity exponent in (Lovejoy et al., 2007) indicates that starting with the theoretical Boligano - Obukhov value 3/5 near the surface, that the exponent increases somewhat with altitude to the value $\approx 0.75$ at 10 - 12 km. Similarly, the humidity (and hence log$\theta_E$ values) may have both horizontal and vertical variations which may account for their high $H_z$ values. We should therefore regard these studies as only first attempts to quantify the stratification.
Table 6: The above uses the aircraft values for the horizontal exponents and then uses estimates of the $C_1$ and $H$ for the 10-12 km altitude level. The $H$'s are about 0.05 higher than the overall vertical average (table 5) and the $C_1$'s are a also a little higher with the exception of the $C_1$ for $h$ which is a little lower. With the exception of the passive scalar (lidar backscatter), error bars are not given for the $C_1$ estimates since they are not reliably estimated (and will be large). For the horizontal $v$, the transverse wind component was used since it was not very coherent with the altitude fluctuations and was considered more reliable.

4. Space-time Cascades:

4.1 Discussion:

4.1.1 The relation between space and time in fluid mechanics:

A basic property of fluid systems is that there exists a relatively well-defined lifetime for structures of a given size, the “eddy turn over time”. These statistical size/duration relations are the basic physics behind the “space-time” or “Stommel” diagrammes, presented in meteorology textbooks as conceptual tools, but which are in practice never empirically calculated. Likewise, although space-time relations are in fact used all the time in meteorological measurements, they are usually implicit rather than explicit, in the form of “rules of thumb”. For example, many automatic digital weather stations average measurements at the fairly arbitrary period of 15 minutes. If the meso-scale gap existed, this might have had some justification, but if there is no gap, how long should the averaging be made? Alternatively, how often should a weather radar scan if the spatial resolution is 1km? If it is 4 km? Conversely if only “climate” time scale (say monthly) estimates are needed what should be the spatial scale of the corresponding maps? In the same vein, in situ measurements are often considered to be “point measurements” i.e. with infinite (or very high) resolutions, but this is misleading since even if they are at points in space, they are never also instantaneous, i.e. they are not points in space-time, and it is their space-time resolutions that are important for their statistics.

4.1.2 The transition from weather to the climate:

Turning our attention to time scales of weeks to months, we may ask what is the distinction between meteorology and the climate? Although meteorology and climatology are increasingly considered to be distinct sciences, the actual boundary between “climate scales” and “meteorological scales” is not clear and there are no universally accepted definitions. It is still hard to improve upon the old adage “the climate is what you expect, the weather is what you get”. For example, the National
Academy of Science essentially accepts this saying in more scientific terms: “Climate is conventionally defined as the long-term statistics of the weather...” proposing only “…to expand the definition of climate to encompass the oceanic and terrestrial spheres as well as chemical components of the atmosphere” (Committee on Radiative Forcing Effects on Climate, 2005).

If there is (at least statistically) a one-to-one relation between space and time scales, we can ask what is the time scale of planetary scale structures? An early response to this was based on the “synoptic maximum” (Kolesnikov and Monin, 1965) which was essentially a drastic change in the scaling observed in the pressure spectrum at around two weeks (≈ 10^6 s) which was interpreted as the time scale of planetary scale structures. Indeed, similar breaks (with similar interpretations) in temporal scaling have been observed in temperature spectra (Lovejoy and Schertzer, 1986) and in spectra of rain rates from gauges (Tessier et al., 1996) see fig. 15 a for the temperature spectrum which (at least for scales less than 2 months) is very close to the IR satellite spectra from MTSAT (see below) and for a straightforward temporal extrapolation of the space-time turbulent “weather” model (fig. 15 b) explained in section 4.5. Although it is not clear how far the plateau extends to low frequencies, in the case of the temperature field, (Lovejoy and Schertzer, 1986) argue on the basis of a nearly four century long series from central England that it finally ends at scales of 300 years or so. Analysing northern hemisphere global temperature spectra, they also showed that the plateau continued out to about five years, apparently rising for the lower frequencies.
Fig. 15a (top): The spectral plateau; reproduced from (Lovejoy and Schertzer, 1986). This is the average of 5 daily temperature series each 6 years long from a station in France. The red reference slope is -2; this is the estimate for $\beta$ using $H = 0.60$ (from table 3) and an intermittency correction $K(2) = 0.2$ (i.e. as predicted if the horizontal and temporal exponents are equal). The faint black reference lines have slopes 0, -5/3. Note the strong annual cycle and the change of about a factor of $10^2$ between the daily variance and the low frequency variance.

Fig. 15b (bottom): The spectrum is well reproduced by the multifractal model (section 4.5) which is simply the extrapolation of the turbulence model to time scales longer than the lifetime of a planetary size eddy (taken here as 16 simulated days). We show the average of 12 simulations of daily temperature, over 90 years, see the details in section 4.5. The straight line has the theoretical slope -1.47 corresponding to $\alpha = 1.8, C_1 = 0.1, H = 1/3$ (this was a simulation for the horizontal wind so the $H$ was a little lower than for $T$, but this only slightly changes the slope at the high frequency “meteorological” part of the spectrum.

Although we argue below that this interpretation is essentially correct, in order to be convincing it must be theoretically based. Let us briefly consider the consequences
of applying the Kolmogorov relation $V \approx \varepsilon^{1/3} L^{1/3}$ - which we have argued holds in the horizontal up to planetary scales $L$ ($V$ is the typical velocity across a structure of size $L$). First, we can estimate the mean energy flux $\varepsilon$ by considering that the mean solar flux absorbed by the earth is $200 \, \text{W/m}^2$ (e.g. (Monin, 1972)). If we distribute this over the troposphere (thickness $\approx 10^4 \, \text{m}$), with an air density $\approx 1 \, \text{Kg/m}^3$, and assume a 2% conversion of energy into kinetic energy ((Palmén, 1959), (Monin, 1972)), then we obtain a typical value $\varepsilon \approx 4 \times 10^{-4} \, \text{m}^2/\text{s}^3$ which is indeed the typical value measured in small scale turbulence ((Brunt, 1939), (Monin, 1972)); (the geometric mean $\varepsilon$ measured in the aircraft legs discussed in section 2 was $4.3 \times 10^{-4} \, \text{m}^2/\text{s}^3$). If we now estimate the large scale $V$ by using the Kolmogorov formula and taking $L = 10^7 \, \text{m}$, then we obtain $V \approx 10^{-15} \, \text{m/s}$ ($\approx 1000 \, \text{km/day}$) which is a reasonable estimate of the typical winds far from the surface. Using this we find the corresponding time scale $L/V \approx 10^6 \, \text{s}$ or roughly two weeks. (Radkevitch et al., 2008) used meteorological analyses over a regional model at scale $6000 \, \text{km}$ to determine the statistical distribution of eddy turnover times finding a mean of $9.5 \times 10^5 \, \text{s}$ with only a narrow dispersion the planetary scale turnover time should be a bit longer. These estimates - which assume that the formula $V \approx \varepsilon^{1/3} L^{1/3}$ holds over huge ranges – turn out to be quite realistic, however they cannot be justified in the framework of the classical isotropic 2D / isotropic 3D model since the latter only assume its validity over the limited small scale isotropic 3D range.

In the next subsection we extend the anisotropic spatial turbulence model discussed in section 3 to anisotropic space-time turbulence (section 4.2); we will see that this analysis justifies the above interpretation quite straightforwardly. In any case, convincing justification of the model requires the systematic establishment of space-time relations over the entire of small to large scales. This is our task in section 4.3, 4.4 where we extend the empirically determination of space-space to space-time diagrammes.

4.1.3 Predictability, forecasting:

Clear knowledge of space-time relations is also needed in forecasting. This is because small scale perturbations grow progressively “polluting” the larger scales via an inverse cascade of errors (Lorenz, 1969). To see how this works, approximate the inverse cascade via a series of cascade steps, each over an octave in scale. The time for the error to propagate from one octave to the next larger one is roughly the corresponding eddy turnover time / lifetime. The overall limits to predictability are obtained by summing over all the octaves from smallest to largest. If we now consider the classical 2-D/3-D paradigm, we find that whereas in 3-D turbulence (dominated by energy fluxes $\varepsilon$) a structure of size $L$ has the lifetime $\tau_\varepsilon = \varepsilon^{-1/3} L^{2/3}$ in 2-D enstrophy cascades it is independent of size $\tau_\varepsilon = \zeta^{-1/3}$ where $\zeta$ is the enstrophy flux. The result is that in 3-D the overall limit is the sum of a converging algebraic series (nearly equal to the turn-over time of the largest structure), whereas in 2-D it can in principle be much larger than this depending on the overall number of octaves in the enstrophy cascade. In our anisotropic cascade model, the horizontal is still dominated by $\varepsilon$ so that the limits to predictability are still roughly equal to the eddy turn over time of the largest eddy, i.e. about 2 weeks. This is indeed the frequently cited predictability limit (see e.g. (Lorenz,
1969)) which is based on statistical closure techniques applied to 3D isotropic turbulence. It should be mentioned that the above picture needs important nuances, since (Schertzer and Lovejoy, 2004) show that in multifractal cascades, the error growth is actually highly intermittent, coming in “puffs”.

4.2 Anisotropic space-time turbulence:

4.2.1 Space-time scale functions:

In sections 2 and 3 we have argued that atmospheric variables including the wind have wide range (anisotropic) scaling statistics. Dimensionally, a velocity is needed to connect space and time and physically the wind advects the fields. Therefore it is hard to avoid the conclusion that spatially scaling fields should also be (anisotropically) scaling in space-time (and hence be the product of space-time cascade processes). While we argue that this is true, space-time scaling is unfortunately somewhat more complicated than pure spatial scaling. At meteorological time scales this is because we must take into account the mean advection of structures and the Galilean invariance of the dynamics. At longer climatological time scales, this is because we consider the statistics of many lifetimes (“eddy-turn-over times”) of structures. We first consider the shorter timescales. This discussion is a summary of a more detailed one in (Lovejoy et al., 2008b).

In order to illustrate the formalism, we shall discuss the example of the horizontal wind $v$. Let us consider the 23/9D model (section 3) in which the energy flux $\varepsilon$ dominates the horizontal and the buoyancy variance flux dominates $\phi$ the vertical so that horizontal wind differences follow:

\[
\Delta v(\Delta x) = \varepsilon^{1/3} \Delta x^{1/3}; \quad a
\]
\[
\Delta v(\Delta y) = \varepsilon^{1/3} \Delta y^{1/3}; \quad b
\]
\[
\Delta v(\Delta z) = \phi^{4/5} \Delta z^{3/5}; \quad c
\]
\[
\Delta v(\Delta t) = \varepsilon^{1/2} \Delta t^{1/2}; \quad d
\]

where $\Delta x$, $\Delta y$, $\Delta z$, $\Delta t$ are the increments in horizontal coordinates, vertical coordinate and time respectively. Equations (20 a-b) describe the real space horizontal Kolomogorov scaling and 20 c the vertical Bolgiano-Obukhov (BO) scaling for the velocity, the equality signs should be understood in the sense that each side of the equation has the same scaling properties. The anisotropic Corrsin-Obukov law for passive scalar advection is obtained by the replacements $v \rightarrow \rho; \ \varepsilon \rightarrow \chi^{3/2} \varepsilon^{-1/2}$ where $\rho$ is the passive scalar density, $\chi$ is the passive scalar variance flux. We have included eq. 20 d which is the result for the pure time evolution in the absence of an overall advection velocity; this is the classical Lagrangian version of the Kolmogorov law.

Following the developments in section 3, we can express the scaling (eqs. 20 a-d) in a single expression valid for any space-time vector displacement $\Delta R = (\Delta x, \Delta y, \Delta z, \Delta t)$ by introducing a scalar function of space-time vectors called the “(space-time) scale function”, denoted $\| \Delta R \|$, which satisfies the fundamental (functional) scale equation:
\[
\left[ \lambda^{-G_s} \Delta R \right] = \lambda^{-1} \left[ \Delta R \right]; \quad G_s = \begin{pmatrix} G_s & 0 & 0 \\ 0 & H_i & 0 \end{pmatrix}; \quad H_i = (1/3)/(1/2) = 2/3 \quad (21)
\]

where \( G_s \) is the 3X3 matrix spatial generator (eq. 16).

Using the space-time scale function, we may now write the space-time generalization of the Kolmogorov law (eq. 20) as:

\[
\Delta \nu(\Delta R) = \epsilon^{1/3} \left[ \Delta R \right]^{1/3} \quad (22)
\]

where the subscripts on the flux indicate the space-time scale over which it is averaged.

The result analogous to that of section 3, the corresponding simple (“canonical”) space-time scale function is:

\[
\left[ \Delta R \right]_{can} = l_s \left( \left( \frac{||\Delta \nu||}{l_s} \right)^2 + \left( \frac{|\Delta \nu|}{\tau_s} \right)^{2/H_i} \right)^{1/2} \quad (23)
\]

(see Marsan et al., 1996)). Where \( \tau_s = \phi^{-1/2} \epsilon^{1/2} \) is the “sphero-time” analogous to the spher-scale \( l_s = \phi^{-3/4} \epsilon^{5/4} \).

### 4.2.2 Advection and Gallilean invariance:

Unfortunately, the above is missing a key ingredient: advection. When studying laboratory turbulence generated by an imposed flow with velocity \( V \) with superposed turbulent fluctuations, (Taylor, 1938) proposed that the turbulence is “frozen” such that the pattern of turbulence blows past the measuring point sufficiently fast so that it doesn’t have time to evolve; i.e. he proposed that the spatial statistics could be obtained from time series by the deterministic transformation \( V \Delta t \rightarrow \Delta x \). While this transformation has been frequently been used in interpreting meteorological series, it can only be properly justified by assuming the existence of a scale separation between small and large scales so that the large scales really do blow the small scale (nearly frozen) structures past the observing point, and we have argued that there is no scale separation in the atmosphere.

However, if we are only interested in the statistical relation between time and space and the system is scaling, then advection can be taken into account using the Gallilean transformation matrix \( A \):

\[
A = \begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (24)
\]

where the mean wind vector has components: \( v = (u, v, w) \). (Schertzer et al., 1997)). The new generator is \( G_{st, adv} = A^{-1} G_s A \) and the scale function \[ \left[ \Delta R \right]_{adv} \] which is symmetric with respect to \( G_{st, adv} \) is:

\[ \left[ \Delta R \right]_{adv} = \left[ A^{-1} \Delta R \right] \]. The canonical advected scale function is therefore:
\[ \|A^{-1} \Delta r\|_{\text{adv,can}} = \left( \frac{\Delta x - u \Delta t}{l_s} \right)^2 + \left( \frac{\Delta y - v \Delta t}{l_s} \right)^2 + \left( \frac{\Delta z - w \Delta t}{l_s} \right)^{2/H_z} + \left( \frac{\Delta \tau}{\tau_x} \right)^{2/H_s} \]  

(25)

Note that since \( D_{st,\text{adv}} = TrG_{st,\text{adv}} = TrA^{-1}G_x A = TrG_{st} = D_{st} \), constant advection does not affect the elliptical dimension (see however below).

### 4.2.3 Advection in the horizontal:

Formula 25 is valid due to the Gallilean invariance of the equations and boundary conditions; it assumes that the advection velocity is essentially constant over the region and independent of scale. We now consider this in more detail. Let us first consider horizontal advection (put \( w = 0 \)). If we apply the formula over a finite region with relatively well-defined mean horizontal velocity, then it should apply (see the discussion in (Lovejoy et al., 2008b)). But what about applying it to very large e.g. global scale regions where the mean velocity is small (if only because of rough north–south symmetry)? However, even if we consider a flow with zero imposed mean horizontal velocity (as argued by (Tennekes, 1975)) in a scaling turbulent regime with \( \Delta \eta \approx \epsilon^{1/3} \), the typical largest eddy (size \( L \)) will have a mean velocity \( V \Delta \eta \approx \epsilon^{1/3} L^{1/3} \) and will survive for the corresponding eddy turn over time \( \tau_{e,L} = L/V = \epsilon^{1/3} L^{2/3} \). In other words, if there is no break in the scaling then structures are advected by the largest structures in the scaling regime. Following the discussion in section 4.1.2, using the typical values \( \epsilon \approx 10^{-4} - 10^{-3} \text{ m}^2 \text{ s}^{-3} \), this implies \( V \approx 10 - 15 \text{ m/s}, T = \tau_{e,L} \approx 10^6 \text{ s} \).

With this estimate of the horizontal velocities to insert in eq. 25, let us compare them to the Lagrangian term \( (\Delta t/\tau_x)^{1/H_z} \) considering only the temporal variations (i.e. take \( \Delta x = \Delta y = \Delta z = 0 \) and taking horizontal axes such that the advection term is \( V \Delta t/l_x \)). By definition, the sphero-time \( \tau_x \) satisfies: \( l_x = \epsilon^{1/3} \tau_x^{1/2} \) and since \( T = V^2/\epsilon \) we see that the condition that the pure temporal evolution term is negligible (i.e. \( V \Delta t/l_x > (\Delta t/\tau_x)^{3/2} \)) is \( \Delta t < T \) so that the term \( (\Delta t/T)^3 \) only becomes important for \( \Delta t \approx 2 \) weeks. However, since the physical size of the eddies with lifetime \( \Delta t = T \) is already the size of the planet (\( L \)), presumably the term ceases to be valid for scales \( \Delta t > T \). However, it may play a modest role in breaking the scaling for \( \Delta t \) comparable to \( T \), i.e. for the transition from weather to climate.

We can now use this information to rewrite the horizontal scale function (eq. 25 with \( w = 0 \)) in terms of \( L, T \) and \( V \) instead of \( l_x, \tau_x \):

\[ \| (\Delta r, \Delta t) \| = \left( \| \Delta r - v \Delta t \|^2 + L^2 T^{-3} \Delta t^3 \right)^{1/2}; \quad T = L/V = L^{2/3} \epsilon^{-1/3} \]  

(26)

(using \( \epsilon_L = L^2/T^3 \)), where \( v \) is a horizontal velocity, \( \Delta r \) a horizontal lag vector and we have used \( H_z = 2/3 \). However, if we are interested in a globally averaged scale function, we can use \( \langle \Delta r \cdot v \rangle = 0 \) (true at least because of rough north/south symmetry), and \( \langle v \cdot v \rangle = V^2 \) so that we obtain:

\[ \| (\Delta r, \Delta t) \| \approx \left( \| \Delta r \|^2 + V^2 \Delta t^2 + \epsilon_L \Delta t^3 \right)^{1/2} \]  

(27)
This formula has a number of interesting features: a) for $\Delta t < T$, (the weather regime) the $\varepsilon \Delta t^3$ term is negligible, the generator is isotropic in $(\Delta x, \Delta y, V\Delta t)$ space, this allows it to account for the observed meteorological scale space-time relations, b) for $\Delta t > T$, (the climate regime) the dominant term is $\varepsilon \Delta t^3$ which shows that the climate depends on the solar energy flux modulated by the clouds and dynamics, c) the averaging – by removing the cross-term $\frac{\overline{\Delta R \cdot \nu}}{}$ - has broken the scaling i.e. whereas eq. 26 is a valid scale function, 27 is not (at least not under linear GSI i.e. with $G$ a matrix, it may be a scale function with respect to a nonlinear extension of GSI). However, the break occurs at scale $L$ where in any case there is a dimensional transition (see section 4.5).

### 4.2.4 The effect of the vertical wind

According to the above, the largest eddies “sweep” the smaller ones so that for time scales less than about 2 weeks we can ignore the pure time development term. This just leaves the horizontal and vertical advection terms $(V\Delta t/l_s$ and $(w\Delta t/l_s)^{1/H_z})$. In order to compare them we must take into account that while the mean horizontal wind across a given part of the earth may be relatively large and well defined (insensitive to the resolution which mostly affects its fluctuations), the same is not true of the vertical wind. When $w$ is averaged over time scale $\Delta t$ (denoted $w_{\Delta t}$) it tends to zero as the region of interest increases in size and with increasing temporal averaging ($\Delta t$), in other words, statistically $w_{\Delta t} \approx \Delta t^{-\gamma}$ where $\gamma$ is a small typically positive exponent (and due to the cascade, presumably random). (Lovejoy et al., 2008b) argues that statistically the net effect of this is to replace $(\Delta t / \tau_s)^{1/H_z}$ with $(\Delta t / \tau_s')^{1/H_z'}$ (and putting the “effective vertical velocity” to zero) we may replace $G_{st}$ with an “effective generator” and effective advection matrix:

$$G_{st, \text{eff}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & H_z' & 0 \\ 0 & 0 & 0 & H_z' \\ \end{pmatrix} ; \quad A_{\text{eff}} = \begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \quad D_{st, \text{eff, advvec}} = \text{Tr}(A_{\text{eff}}^2 G_{st, \text{eff}} A_{\text{eff}}) = \text{Tr} G_{st, \text{eff}} = 2 + H_z + H_z'$$

with corresponding “effective scale function”:

$$\left[ \overline{\Delta R}_{\text{advvec, eff, com}} \right] = \left[ A^{-1} \overline{\Delta R}_{\text{eff, com}} \right] = l_s \left[ \frac{\Delta x - u \Delta t}{l_s} \right]^2 + \left[ \frac{\Delta y - v \Delta t}{l_s} \right]^2 + \left( \frac{\Delta z}{l_z} \right)^{2/H_z} + \left( \frac{\Delta t}{\tau'_s} \right)^{2/H_z'} \right]^{1/2}$$

The exponent $H_z'$ and the value $\tau'_s$'s depend on the exact (scaling) statistics of the vertical wind which are not known, although (Radkevitch et al., 2008) finds empirically using lidar $(z,t)$ sections that $H_z' \approx 0.7$ and $\tau'_s$ has a large variability but is somewhat larger than the spheroc-time $\tau_s$ (which is also highly variable see the discussion in (Lovejoy et al., 2008b)). Using meteorological analyses, (Radkevitch et al., 2008) show that nevertheless the pure temporal development will still dominate at large enough time scales.
4.3 Evidence for vertical space-time scaling from lidar

4.3.1 Spectra

If over an experimental region, $\tau_s$ is large enough, and the local horizontal wind is small enough, then the $\left(\frac{\Delta t}{\tau_s'}\right)^{1/H_t}$ terms may be dominant; below using lidar $(z,t)$ sections, we show that this is indeed the case. When this regime is dominant, passive scalars will have temporal scaling exponents $H_t = (1/3)/H_t' \approx 0.48$ corresponding to spectra with exponents $\beta = 1+2H \approx 2$ which is a bit steeper than the usual Corrsin-Obukhov $\beta = 5/3$. (Lovejoy et al., 2008b), discusses various reports in the literature of $\beta \approx 2$ exponents, this could be their origin. We summarize a small part of the analyses of (Radkevitch et al., 2007; Radkevitch et al., 2008), which uses lidar backscatter from stationary, vertical pointing lidar using very similar data to that discussed in section 3.

Fig. 16 a, b shows the 1-D spectra in the vertical and in time. In fig. 16 a we can clearly see the Bolgiano - Obukhov (BO) scaling in the vertical. Some deviations from the theoretically predicted slope are due to problems with lidar attenuation corrections (see especially the second spectrum from the top in Fig. 16 a). Other technical limitations including in particular attenuation and inadequate dynamic range are discussed in (Radkevitch et al., 2008). As far as we can tell, the cirrus and aerosol scalings are the same and both are compatible with the anisotropic extension of the Corrsin - Obukhov law. We can see that the BO value $= 11/5$ works well over a wide range of scale.

The results of 1-D spectral analyses in time are represented by Fig. 16 b; we see that there are three cases with $\beta_t \approx 2$ and three other cases showing $\beta_t \approx 5/3$ (actually, out of a total of 15 $(z, t)$ sections analyzed these were the only ones which showed clear evidence of $\approx \omega^2$ spectra). Ignoring intermittency, these spectra correspond to $H_t = (\beta_t - 1)/2 = 1/2, 1/3$. Since in the vertical we found $\beta_v \approx 11/5$, we have $H_t = (\beta_v - 1)/2 = 3/5$, and thus $H = H_v H_t \approx 2/3, 5/9$ for $H_t = 1/2, 1/3$ respectively. As we have argued, although it is very unlikely to observe pure temporal development from time scales below around $10^4-10^5$ s, it is quite possible to occasionally observe vertical wind dominance over horizontal wind.
Fig. 16 a: 1-D space (vertical spectra as a function of vertical wavenumber $k_z \ (m^{-1})$ for vertical – time ($(z,t)$ sections) datasets (offset in the vertical for clarity). The top two are cirrus, the rest are aerosol backscatter. The dashed reference lines have slopes -11/5 corresponding to $H_v =3/5$ (ignoring intermittency). Top to bottom experiments E0530, E0602, E0603, E616, L0807, L0808. Reproduced from (Radkevitch et al., 2008).
Fig. 16 b: 1-D time spectra as functions of frequency $\omega (s^{-1})$ for the same $(z,t)$ sections as fig. 16 a (note the order is not the same). Top to bottom experiments E0602, E0603, L0807, E0616, L0808, E0530. The top three reference lines have slope -2, the bottom 3 have reference lines -5/3. Reproduced from (Radkevitch et al., 2008).
Fig. 16 c: The is a space-time (vertical/time) diagramme obtained from the first order structure functions of 3 lidar time series at 1 s (red) and 2 s (blue, green) resolutions. At the largest scales, the statistics are not as good. We see that that troposphere thickness (which corresponds roughly to planetary sizes in the horizontal) has a time scale of several weeks to a month, see section 4.1.2. Assuming that \( l_s = 1 \) m, the top line corresponds to \( v = 60 \) m/s, The bottom line to 5 m/s. If instead \( l_s = 10 \) cm, top line implies 400 m/s, bottom to 30 m/s. This is estimated using the formula: 
\[ v \Delta t / l_s = \left( \Delta z / l_s \right)^{1/H_z}. \]
We thank A. Radkevitch for help with this analysis.

**4.3.2 Space-time diagrammes from lidar**

Finally, following the construction of the “space-space” diagramme showing the relation of horizontal to vertical scales (section 3), we can use first order structure functions to determine “space-time” or “Stommel” diagrammes from the lidar \((z,t)\) data. To do this, for a given \( \Delta t \), the corresponding \( \Delta z(\Delta t) \) is determined by the solution of the implicit equation for the first order structure functions 
\[ \langle |\Delta B(\Delta z)| \rangle = \langle |\Delta B(\Delta t)| \rangle \]  
\((B\) is the lidar backscatter ratio\)). For three of the longer \((z,t)\) sections from section 4.3.1, the results are shown in fig. 16 c, we see that the data follow reasonably accurately the theoretical curve (assuming horizontal wind dominated temporal statistics and \( H_z = 5/9 \)). In addition, if the sphero-scale is assume to be 1 m (roughly what was determined for the vertical section data in fig. 13 c), then we find horizontal wind in the range 5 – 60 m/s which is quite reasonable. We also see that the space-time diagramme gives direct evidence that the top of the troposphere (10 km) corresponds to the outer time scale \( \approx 2 \) weeks.
4.4 Evidence from satellite radiances: horizontal – time analyses

4.4.1 MTSAT thermal IR:

The lidar data considered in the previous section allows us to determine the space-time properties of vertical spatial scales up to kilometers and time scales of seconds to hours. If we seek to study the relation of horizontal to temporal scales, then satellite data is ideal. In section 2.3 we used TRMM and MTSAT satellite radiances to examine the large horizontal scale statistical properties of passive and active microwaves, visible and infra red radiation. We can also use the same methodology – the systematic degradation of fluxes estimated from absolute gradients – to degrade the same data in the time domain. For this purpose, the TRMM satellite has the disadvantage of having an average 2 – 4 day return time (depending on the wavelength and location) so that it has poor temporal resolution. Let us therefore first examine the MTSAT thermal IR data with 1 hour resolution. Since we find that 1 hour corresponds to about $\approx 30 \, km$, we degraded the nominally $5 \, km$ resolution data to $30 \, km$; fig. 17 a shows the result using 2 months of data (1440 images each between $30^\circ \, N$ and $40^\circ \, S$ by $130^\circ$ east-west). The temporal statistics are shifted left - right on the log-log plot so that they coincide almost exactly with the east-west spatial statistics; this implies that the spatial statistics are identical to temporal statistics if time is transformed into space with a velocity of $\approx 900 \, km/day$ ($\approx 10 \, m/s$) which is similar to that determined earlier from the mean energy flux and from the lidar space-time diagramme.

We can now use the normalized flux moments to determine the space-time diagramme, here the $\Delta z$, $\Delta t$ relation is determined by $M_q(\Delta t) = M_q(\Delta z)$ where $M_q$ is the normalized $q^{th}$ order moment. Note that in principle, we could obtain a different space-time relation for low and high order moments $q$ (corresponding to different relations for weak and intense events/structures). However inspection of fig. 17 a shows that since the time and space moments for all $q$ are very similar, that there will be only very small differences, fig. 17 b shows the corresponding space-time diagramme. Comparing this with the mean east-west/north-south statistics (fig. 7 c) we see that the latter is very similar to the TRMM thermal IR (at typically a direction of $\approx 45^\circ$ to the equator); it has excellent scaling. However, taken individually the north-south and east-west scalings are not so good at the largest scales, but nevertheless, time and (east-west) space are virtually identical including the deviations from scaling at large scales. The result is that the east- west space –time diagramme is nearly perfectly scaling over the entire range.
Fig. 17 a: An intercomparison of $\log_{10} M_q(\Delta t)$ for east west radiance fluxes (blue) and time (pink) for $q = 0.4, 1.2, 2, 2.8$. $\lambda$ is defined with respect to a time scale of 2 months for the temporal analyses. The spatial $\log_{10} M_q(\Delta x)$ has been shifted so as to superpose as closely as possible on the $\log_{10} M_q(\Delta t)$ curves. The corresponding speed is $\approx 900 \text{ km/day}$ ($10 \text{ m/s}$) and the outer cascade scale is $\approx 40 \text{ days in time, } \approx 35000 \text{ km in space}$. The deviations from scaling become important at about $\approx 5000 \text{ km}$ or about $\approx 6 \text{ days}$. Compare this with the nearly perfectly scaling fig. 7 c which is the geometric mean of the east west above with the north-south analysis (from (Pinel, 2009)).
4.4.2 TRMM thermal IR:

Returning to the TRMM data at a 12 hour resolution we averaged one year (≈5300 orbits) of the thermal IR over 100X100 km grids. The results are shown in fig. 18 a. Note that the below 2 days the statistics are poor since only a fraction of the 100X100 “pixels” are visited at time intervals less than this. We see that the plot can be divided into three regions. Up to about 10 days, the moments are relatively linear as expected from space-time multiplicative cascade processes. If we extrapolate the lines to larger scales, they cross at about 25 days so that the variability at less than 10 days is accurately that which would result from multiplicative cascade starting at 25 days. At scales larger than this they have yet another behaviour which we discuss later. To aid in the interpretation, Fig. 18 b shows the superposition of the east-west spatial analysis of the same data. We see that although the resolution is much lower and hence the scaling region much shorter – that up to about 10,000 km and 10 - 15 days, the fit is comparable to that of the MTSAT although the velocity is somewhat smaller, ≈ 400 km/day. According to our theoretical discussion about space-time statistics in anisotropic turbulence, it is natural to identify the scales where the space-time relation is linear with the weather, and those at scales larger than the outer temporal cascade scale with the climate, with an intermediate “transition” region between the two; we return to this in section 4.5.
Fig. 18 a: The normalized moments of the TRMM thermal IR data averaged over 100x100 km pixels at 12 hour resolution from 5300 orbits (1 year corresponding to $\lambda = 1$). The long time variability has been fit to a cascade with outer scale at 1100 days, but it is not clear that this is a good model (see section 4.5).
Fig. 18 b: The same normalized moments of the TRMM thermal IR data as fig. 18 a but with temporal and spatial moments superposed corresponding to a velocity of 400 km/day. The pink is the temporal analysis (from fig 18a), the blue is the east–west spatial analysis (corresponding to fig. 7 b except that the analysis is not along orbit, and is at lower resolution).

4.4.3 Numerical models:

We can also perform scale by scale temporal analyses of the numerical model fluxes discussed in section 2.2 and compare them with the corresponding spatial analyses. Fig. 19 a shows typical results for the ERA 40 reanalyses (the full results are in (Stolle, 2009)). The 700 mb east-west field was chosen (analysed in the east-west direction) since it can be most conveniently compared to the thermal IR (fig. 18 a; 700 mb is roughly the level where a “typical” thermal IR signal originates). We see that it is very similar both qualitatively and quantitatively to the thermal IR, with only slightly different external cascade scale (40 days), and outer meteorological scale (8 days). The best space-time transformation corresponds to a velocity of ≈ 400 km/day which is a bit smaller that of the IR imagery. (Stolle, 2009) makes the comparable analyses for the GEM and GFS model; the results are sufficiently similar that we do not repeat them here.

We can also construct the corresponding model space-time diagrammes, fig. 19 b (also using the $M_q$’s). In this case, we estimate it for low medium and high order moments to see whether there is a significant difference in the space–time relations for weak, medium and intense events. From the figure, we see that the differences are small
except for some deviations for the weak events at short distances/times. We can now compare these results with the MTSAT and TRMM IR analyses, see table 7.

<table>
<thead>
<tr>
<th></th>
<th>Weather (days)</th>
<th>Climate (days)</th>
<th>Effective outer scale (km)</th>
<th>Speed (km/day)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MTSAT IR</td>
<td>6</td>
<td>40</td>
<td>35000</td>
<td>900</td>
</tr>
<tr>
<td>TRMM IR</td>
<td>10</td>
<td>25</td>
<td>12500</td>
<td>400</td>
</tr>
<tr>
<td>ERA 40 (east-west wind)</td>
<td>8</td>
<td>40</td>
<td>12000</td>
<td>400</td>
</tr>
</tbody>
</table>

Table 7: An intercomparison of the various time and space scales estimated from the satellite imagery and the ERA 40 reanalyses. Whereas the climate time scale and effective outer spatial scale are determined from the point where the straight lines in the normalized moments ($M_q$) cross the weather scale is a somewhat subjectively estimated from the point at which the temporal scaling becomes poor. The weather-climate transition regime lies between these two scales.

Fig. 19 a: ERA 40 at 700 mb, east-west wind, three years of daily reanalyses were used. Reproduced from (Stolle, 2009).
4.5 The climate as a dimensional transition from the weather: extending the cascade model to long time periods

4.5.1 Discussion:

Considering just the horizontal and time, we have argued that the velocity scale linking time and space should be precisely the typical velocity of the largest eddy $V$ which is determined by the external spatial scale $L$ (the size of the planet) and the driving global mean energy flux $\varepsilon_L$ (itself determined by the solar radiation modulated by the clouds and dynamics; $V = \varepsilon_L^{1/3} L^{1/3}$ and $T = L/V$). In other words, $L$, $V$, $T$ are determined from basic principles, they are not simply adjustable model parameters. Considering the scale function for a planetary scale region with small overall mean velocity (but not small rms velocity $V$), we argued that for scales $l < L$ and $\tau < T$, it was essentially isotropic (eq. 27; in nondimensional coordinates so that $V = 1$, see below).

While we have spent effort testing the prediction of the basic model for scales $l < L$ and $\tau < T$, we have not examined the behaviour of the model for scales $\tau > T$: can the same model account for the weather–climate transition, and to what extent can it account for the climate regime (e.g. the spectral plateau)? In other words, what are the limits of the model, at what scales does it finally break down?

To attempt to answer this question, we turn to explicit space-time stochastic cascade models. We first recall the basic features of (continuous in scale) cascade
model for the turbulent flux $\varepsilon$. First, since it is assumed to be a multiplicative process, it can be expressed in terms of the exponential of an additive generator $\Gamma$:

$$\varepsilon(r,t) = \langle \varepsilon \rangle e^{\Gamma(r,t)}$$  \hspace{1cm} (30)$$

where $\Gamma$ is the (dimensionless) generator. If we assume that the basic statistics are translationally invariant in space-time (statistically homogeneous, statistically stationary), then $G$ is given by a convolution between a basic noise $\gamma(r,t)$ (independent, identically distributed), and $g(r,t)$ is a Green’s function (a deterministic weighting function that correlates them over (potentially) large space-time distances):

$$\Gamma(r,t) = \gamma(r,t) \ast g(r,t)$$  \hspace{1cm} (31)$$

For the stable and attractive processes (leading to universal multifractals), $\gamma(r,t)$ is taken as a unit Levy noise, index $\alpha$, i.e. whose second characteristic function is $\log \langle e^{\gamma} \rangle = q^{\alpha} / (\alpha - 1)$. In addition, $g$ must have a particular form:

$$g(r,t) = n_d \epsilon_1^{1/\alpha} \left[ \left( (r,t) \right) \right]^{-D/\alpha}$$  \hspace{1cm} (32)$$

with the singularity cutoff at the inner, dissipation scale and $D$ the dimension of space time (the trace of the scale generator $G$ for isotropic space-time, $n_d$ is a normalization constant, $\epsilon_1$ the intermittency parameter of the mean intermittency, $\alpha$ the Levy index of the (extremal) uncorrelated space-time unit amplitude Levy noise $\gamma(r,t)$ (see (Schertzer and Lovejoy, 1987) for the basic model, (Marsan et al., 1996) for the extension to causal space-time processes, (Lovejoy et al., 2008b) for the extension to turbulence driven waves, and (Lovejoy and Schertzer, 2009a) for a technical update on numerical issues. We have used a space-time scale function nondimensionalized by $L, T$.

From the flux, the observable $v$ (e.g. a horizontal wind component) can be obtained whose statistics obey $\text{eqs. 20, if we take:}$$

$$v(r,t) = \varepsilon(r,t)^{1/3} L^{1/3} \left[ \left( (r,t) \right) \right]^{(D-H)}$$  \hspace{1cm} (33)$$

In eqs. 30, 31, we have ignored the issue of causality which can be taken into account with the use of a Heaviside function in the convolutions ((Marsan et al., 1996)); for our present purposes, this complication may be ignored; eq. 33 for the observable is called the “Fractionally Integrated Flux model”.

### 4.5.2 A simple model of $(x, t)$ sections of the weather, the transition and the climate:

We will now consider the consequences of assuming that a multiplicative model of the type defined by $\text{eqs. 30, 31}$ holds for climate scales $\tau >> T$. In order to understand the basic features of the weather, the transition and the climate we can study restrict our attention to a $D = 2$, $(x - t)$ section of the full $(x, y, z, t)$ model. The spectrum of such a model was shown in figure 15 b, and in fig. 20 a, we show the corresponding normalized moments. If we rewrite eqs. 30, 31 nondimensionalizing $x$ with $L$ and $t$ with $T$, then we obtain for the generator $\Gamma(x,t) = \log (\varepsilon(x,t) / \langle \varepsilon \rangle)$:

$$\Gamma(r,t) = \int_{A^1} \int_{A^1} \gamma(r-r',t-t') g(r',t') dr' dt' + \int_{A^1} \int_{A^1} \gamma(r-r'-t-t') g(r',t') dr' dt'$$  \hspace{1cm} (34)$$
$\Lambda = L / L_s = T / T_i$ is the total range of meteorological scales and $T_o > T$ is the overall outer scale of the weather/climate process. Denoting the first term by $\Gamma_w(x,t)$ “w” for “weather” and the second climate scale term $\Gamma_c(t)$ (“c” for “climate”) then for $T_o >> T$, we have approximately:

$$
\Gamma(x,t) \approx \Gamma_w(x,t) + \Gamma_c(t)
$$

$$
\Gamma_w(x,t) = \int_{S_\Lambda} \gamma(x - x', t - t') g(x', t') dx' dt'
$$

$$
\Gamma_c(t) = \int_1^{T_o/T} \tilde{\gamma}(t-t') g(0,t') dt'
$$

where $\tilde{\gamma}(t-t')$ is a spatially integrated Levy noise and $S_\Lambda$ is the quarter unit circle in $(x,t)$ space with the quarter circle around the origin of radius $\Lambda^{-1}$ removed (or squares, for our present purposes the difference is unimportant). The approximation in eq. 35 consists in assuming $g(x,t) \approx g(0,t); \quad t >> x$. $\Gamma_w(x,t)$ is a 2 D (space-time) integral corresponds to the contribution to the variability from the weather regime ($t < 1, x < 1$), and the second $\Gamma_c(x,t)$ is a 1 D (purely) temporal contribution due to the climate regime. This is a kind of “dimensional transition” between weather and climate processes.

![Graph](image_url)

Fig. 20a: The result of averaging over 12 realizations of the $(x, t)$ section of the cascade weather - climate model on a $2^4 \times 2^{15}$ point grid (discussed in the text) with $C_1 = 0.1$, $\alpha = 1.8$, $H = 1/3$. The resolution is taken as 1 day in time, $\approx 1200$ km in space ($1/16$ of $T$,$1/16$ of $L$ respectively, so that according to the analysis, the model temporal outer scale
has been increased a bit with respect to the theoretical $T$). The curves are for the moments $q = 0.1, 0.2, \ldots 1.9$, the spatial and temporal analyses are shown superposed.

Fig. 20 b: This compares the moments $M$ normalized by $K(q)$ with $\alpha = 1.8, C_1=1$ for a 70 year time series (bottom, coloured) and a single pixel of the space-time simulation presented in fig. 15 b, 20 a and for $q = 0, 0.1, 0.2, \ldots 2$ (normalized with $\alpha = 1.8, C_1 = 2$). The collapse of the curves for different $q$ to a single line is evidence that the random variable $\Gamma = \log \varepsilon$ is a Levy with the corresponding index. For the largest scales (smallest $\lambda$), the statistics are too poor to judge ($\lambda = 1$ corresponds to 70 years, the lowest factor of 3 of the range is not shown due to the large fluctuations). The data actually collapses a bit better than the simulation, although this is probably a consequence of analyzing a single realization in each case.

This separation into independent additive weather and climate terms with correlated noises integrated over spaces of different effective dimensions is responsible for the statistical difference between weather and climate, at the level of the fluxes it means that the climate process modulates the weather process at the larger time scales:

$$
\varepsilon_{\Lambda,T_0/T}(x,t) = e^{\Gamma_w(x,t)+\Gamma_c(x,t)} = \varepsilon_{\Lambda,w}(x,t)\varepsilon_{T_0/T,c}(t)
$$

with $\varepsilon_{\Lambda,w}(x,t)$, having the high frequency variability, $\varepsilon_{T_0/T,c}(t)$ the low frequency. The generic result is a “dimensional transition” in the form of a spectral plateau which we investigate in more detail in (Lovejoy and Schertzer, 2009b), see also fig. 15b.

To understand the statistics of the model we can calculate the second characteristic functions of $\log \varepsilon = \log \langle e^{\alpha T} \rangle$; we can use the general formulae (valid of the i.i.d. Levy noise $g$ and deterministic function and any positive function $f$):
\[ f = \int \gamma(s) g(s) ds; \quad \log \langle e^{q_t} \rangle = \frac{q^\alpha}{\alpha - 1} \int g^\alpha(s) ds \]  

We find:

\[
\langle e^q \rangle_{\lambda,w} \approx e^{-\frac{q^\alpha}{\alpha - 1} \int g^\alpha(x',x') dx' dx''}
\]

\[
\langle e^q \rangle_{T_0/T,\varepsilon} \approx e^{-\frac{q^\alpha}{\alpha - 1} \int g^\alpha(0,x') dx'}
\]

To see how the weather term results in a multiscaling power law we use the scale function appropriate for a mean \( \langle \gamma \rangle \approx 0 \) but \( \langle \gamma^2 \rangle = \nu^2 \), (eq. 27) and nondimensionalizing \( x, t \) we have:

\[
g(x,t) = n_D C_1^{1/\alpha} \left( \frac{(x,t)}{D^{1/\alpha}} \right)^{-D/\alpha} = n_D C_1^{1/\alpha} \left( x^2 + t^2 + t^3 \right)^{-1/\alpha}
\]

\[
= n_D C_1^{1/\alpha} \left( x^2 + t^2 \right)^{-1/\alpha}; \quad t < 1
\]

where we have used \( D = 2 \) and (the cubic - pure time development term - is only significant near \( t = 1 \)). If we now use polar coordinates \( r^2 = x^2 + t^2 \), \( dxdt = rdrd\theta \), we see that choosing \( n_D = (\pi/2)^{1/\alpha} \) the integral (over a quarter circular \( S_\alpha \)) in expression for \( \varepsilon_w \) yields the required log behaviour. In addition, normalizing \( \varepsilon_w \) by dividing by the mean so that \( \langle \varepsilon_{w,\varepsilon} \rangle = 1 \), we obtain \( \langle e^q \rangle_{\lambda,w} \approx \Lambda \psi(q) \) with \( \psi(q) = C_1(q^{1/\alpha} - q)/(\alpha - 1) \).

The key point about the weather – climate transition is thus the dimensional transition; the difference in the dimensions of space over which the integrals in the weather and climate terms of eqs. 35, 38 are carried out. In order for the climate regime to display multiscaling, we require \( g(0,t) \approx t^{1/\alpha} \) for \( t > 1 \) whereas the extrapolation of the weather \( g(x,t) \), yields \( g(0,t) \approx t^{-2/\alpha} \), which falls off too quickly. In other words, unless the weather noise becomes substantially more correlated (i.e. the exponent for \( g \) becomes lower for \( t > 1 \)) for the climate regime, the correlations will be relatively short ranged leading to the spectral-plateau white noise - like regime of fig. 15.

In summary, this model predicts the following for the fluxes:

a) the flux statistics factorize into a weather and climate contribution:

\[
\varepsilon(x,t) \approx e^{\varepsilon_w(x,t) + \varepsilon_c(x,t)} = \varepsilon_w(x,t) \varepsilon_c(t)
\]

with \( \varepsilon_w \) having the high frequency variability, \( \varepsilon_c \) the low frequency. The generic result is a “dimensional transition” in the form of a spectral plateau.

b) Since weighted sums of independent Levy variables are still Levy variables, this models predict that the climate variability \( \varepsilon_c(t) \) (estimated by space-time averaging over the weather scales has the same (Log Levy) form at time scales much larger than \( T \). In other words, a prediction that follows from simply eq. 30, 31 (i.e. the multiplicative nature of the process), is that

\[
\left[ \log \left( \langle \varepsilon_{\lambda}^q \rangle / \langle \varepsilon_{\lambda} \rangle^q \right) \right] / K(q) \approx F(\lambda)
\]

where \( F \) is a function only of \( \lambda \), not \( q \). Using a 70 year series of daily mean temperatures from Ohio, we empirically test this (fig. 20); it seems that this prediction holds out to at least scales of
several hundred days (after which the data are too noisy), giving direct evidence that the multiplicative model applies to climate scales.

c) If we consider the observable $v = e \ast g$, then we again have a separation of the convolution into a space-time integral over the weather scales and (approximately) a time only integral for the climate scales, we again have a dimensional transition at $T$.

The model shows that at least some of the statistical features of the transitional and climate regimes – particularly the white noise like spectral plateau - are simply long time ($> T$) consequences of the same basic fluid physics that describes the weather regime at scales ($< T$). A future task is to see how far we can take the model before it breaks down. On the basis of that hemispheric averaged temperatures (Lovejoy and Schertzer, 1986) have empirically shown that the spectral plateau continues only out to about 5 – 10 years before low frequency variability which they argue is once again scaling - becomes important.

## 5. Conclusions

### 5.1 Outstanding issues:

We have proposed a synthesis of state-of-the-art data and nonlinear theory which potentially unifies atmospheric dynamics over huge ranges of space-time scales. However, in many respects this picture is still tentative; below we list just a few of the outstanding scientific issues that it raises.

1. **The relevant fluxes: their physical nature**

   We have criticised the classical turbulence approach for deciding *a priori* which turbulent fluxes are relevant, and this coupled with additional *a priori* isotropy assumptions. Indeed, the (occasionally spectacular) confirmations of the predictions of cascades (e.g. fig. 7) were possible because we only made use of the weaker assumption of scaling - which in any case we tested - and we did so without the need to assume in advance the physical nature of the fluxes.

   We now need to investigate this further. On the one hand using high quality space-time remotely sensed data sets - or outputs from numerical models – we can use the classical techniques of turbulence theory to study fluxes in fourier space i.e. the wavevector/frequency domain. Since the turbulent fluxes are fourier space fluxes (e.g. large to small scale energy transfers), this is the natural framework for doing this. On the other hand, we must try and clarify the physical nature of the cascading fluxes, and eventually their inter-relations, including basic questions such as how many different cascades there are and as well as the nature of their inter-relations, inter-dependencies.

2. **The robustness of the exponents:**

   Anisotropic cascades involve two qualitatively different sets of exponents: the first characterize the change in the statistics of the fluctuations with scale, $(H, C_1, \alpha)$ the second, the exponents that define the notion of scale itself $(G)$. Each is characterized by a (mathematical) group and its generator. From a theoretical point of view, the difficulty is that without knowing $G$, we can’t rigorously define the scales and hence determine the statistical exponents $H, C_1, \alpha$. The solution adopted in the empirical analyses was to
take advantage of two rough empirical facts a) that in the horizontal, the anisotropy was not so strong: or at least not strong enough to prevent most of its effects being washed out by using isotropic analysis techniques (such as averaging over wave vector directions to obtain isotropic spectra), and b) that the vertical eigenvector of $G$ is roughly perpendicular to the horizontal direction. These two properties allowed us to effectively avoid the very difficult estimation of off-diagonal elements of $G$ (Lewis et al., 1999); although these are presumably important for cloud and other morphologies).

Bearing in mind this need to “bootstrap” our analyses by first guessing an appropriate definition of scale ($G$), and then using it to estimate the statistics ($H, C_1, \alpha$) we must then consider the robustness of the parameter estimates. This has two aspects: first there must be attempts at reproducing the existing exponent estimates with other instruments, other methodologies, different analysis techniques. This is especially true of in situ measurements which typically have nontrivial problems (section 2). Second, we mentioned that there are already indications that there appear to be some systematic variation of exponents with altitude (especially the horizontal wind); and why not with latitude? In principle such spatial variations can be accounted for in a scaling cascade framework, but at the price of introducing a nonlinear generator $G$.

3. Scale functions:

The justification of generalized scale invariance is that one should not impose $a$ priori notions of scale (such as isotropic metrics), one should rather allow them to be determined by the nonlinear dynamics; the appropriate scale notions are expected to be “emergent quantities” determined by the turbulent fluxes. This idea is similar to general relativity where the distribution of matter and energy determine the distance notion, the metric, although here, the scale function need not be a metric). In the simplest model, there would be a single space-time scale function valid for all the atmospheric fields. Although this idea is seductive there are indications that things are more complicated; for example our inter-comparison of the empirical stratification parameter $H_z$ (table 6) suggests that there may be at least two scale functions needed – one for the humidity dominated fields (the humidity and equivalent potential temperature) and one for the rest: passive scalar, temperature, pressure and potential temperature with the horizontal velocity field being anomalously low (and hence possibly requiring yet a third scale function).

4. The vertical velocity:

Absent from our discussion was the role of the vertical velocity. This may ultimately provide the key for answering many of the above questions, yet it remains elusive because it is so hard to accurately measure (or model!). Experimental campaigns attempting to accurately measure it’s statistical scaling properties (while avoiding any isotropy assumptions!) are urgently needed.

5. The weather-climate transition and the limits of the model:

We have given evidence that the cascade picture extends up to planetary scales and to time scales of a week or more. We have argued that the observed weather-climate transition is in fact a generic consequence of the basic multiplicative cascade model; that it follows because at long time scales, the spatial correlations are essentially
cut-off by the finite size of the earth. We even gave evidence that at least into the beginning of the climate range (beyond the scaling regime, up to hundreds of days) that the model may still apply (fig. 20). This would mean that at least up to these scales that climate variability is still fundamentally of the same type as weather variability. It is therefore important to determine its real temporal outer limits. The validity of the model - even over just part of the climate range - would be important in improving our understanding of natural climate variability, and hence our ability to discern anthropogenic effects.

5.2 Applications:

Atmospheric science labors under the misapprehension that its basic science issues have long been settled and that its task is limited to the application of known laws - albeit helped by ever larger quantities of data themselves processed in ever more powerful computers exploiting ever more sophisticated algorithms. Atmospheric science is thus increasingly being reduced to the application of techniques, to the development of “products”. We have already argued that a consequence of an undervalued theoretical component is the failure to profit from the technological manna itself to reach a consensus on the basic scale by scale properties. Unfortunately in many areas we have reached the point where even the expenditure of vast resources on urgent applications have been unable to overcome longstanding problems, and often these problems are linked precisely to extreme variability, heterogeneity and space-time resolution issues.

Conversely, if the new synthesis is even approximately correct, it will open up new vistas for applications, many of which are of as yet difficult to discern. Below we list a few of the more immediate ones.

1. The development of resolution independent remote sensing algorithms, resolution independent objective analyses:

The empirical analyses summarized in earlier sections directly demonstrate the strong scale dependencies of many atmospheric fields, showing that they depend in a power law manner on the space-time scales over which they are measured. While remotely sensed fields are radiances - in themselves usually of limited interest - they are often used to determine surrogates for dynamic and thermodynamic fields (e.g. satellite “products”). Well-known examples include radar estimates of rain rates or passive microwave estimates of temperature and humidity profiles. In many cases, considerable effort is exerted to calibrate (“validate”) the algorithm using in situ data. This leads to two resolution dependent difficulties: first that the space-time resolution of the calibration data is not equivalent to those of the radiances, and second that even if the radiance based surrogate is well calibrated at the calibration scale, that may not be true at any other scale. Yet there is nothing special about the surrogate resolution beyond the fact that it may be the best available with existing technology; it is essentially a subjective criterion. A typical symptom of this implicit resolution problem is that when new satellites with improved resolutions become available, that the (sometimes numerous) algorithm/calibration constants have to seriously revised.
On the contrary, if we use the space-time scaling exponents to characterize the resolution dependencies, then these can be used as the basis for developing new scale/resolution invariant techniques (Lovejoy et al., 2001), possibly including resolution-independent Bayesian techniques. As an example, estimates of the earth’s radiation budget should be revisited taking into account the cascade scaling of the radiances discussed in section 2.3. The failure of current budget estimates to take these strong resolution dependencies into account implies the likely presence of biases.

2. Stochastic Forecasting, stochastic parametrisation:

In the introduction, we mentioned the ongoing ensemble forecasting revolution, the fact that today meteorologists’ goal is increasingly a stochastic one: to predict the possible states of the atmosphere as well as their corresponding probabilities of occurrence. However, the current ensemble forecasting technique is essentially a deterministic hybrid which is indirect and problematic on several counts. The main difficulties are a) that it is based on a deterministic framework for the initial objective analysis – which uses statistics only to describe measurement errors - and not the fields themselves – and b) which assumes that the fields evolve according to deterministic nonlinear partial differential equations. While deterministic assumptions may be appropriate for descriptions and models at the dissipation scale, stochastic ones are more appropriate at lower space-time resolution (if only because an infinite number of different dissipation scale fields give rise to the same low resolution fields). The reason that ensemble forecasts are ultimately stochastic is that one attempts to deduce an initial probability distribution of atmospheric states at \( t = 0 \) (“ensemble breeding”), and then the model maps this initial distribution onto a future distribution. In this approach, even the final step – the analysis of the members of the forecast ensemble is nontrivial.

In contrast, the cascades have the important advantage of (potentially) providing a consistent and natural stochastic initial distribution followed by an optimum stochastic forecast. First, the initial state of the atmosphere can be specified in terms of probabilities conditioned by the observations (at the appropriate space-time resolutions). They can then (in principle) take this initial conditional probability distribution and give the (theoretically optimum) stochastic forecast, i.e. the conditional expectations or conditional probabilities of the fields at future times, and this at whatever space-time resolutions are required. While this purely stochastic forecast procedure has yet to be proven in practice, it has the additional advantage that it can in principle (statistically) take into account as wide a range of scales as necessary rather than the current \( 10^2 - 10^3 \). At present, its main limitations are that it has mostly been developed for handling individual scalar fields; generalizations are needed for vector cascades to describe the evolution of atmospheric state vectors (Schertzer and Lovejoy, 1995).

While pure stochastic forecasts are not likely any time soon (except perhaps for nowcasting), it should be soon possibly to apply cascade models to “stochastic parametrisation” of the type introduced by (Palmer, 2001). Stochastic parametrisation is an attempt to increase the variability of the deterministic forecasts so as to increase the rate that the different members of the ensemble diverge from each other. It does this by introducing random numbers at each pixel, representing the random effect of the unmodeled sub grid scales. While at present this is done on a fairly ad hoc basis, the
finding that the stochastic structure of the models can be described by cascades promises to put this on a more sound theoretical basis.

3. Distinguishing natural from anthropogenic variability and the problem of outliers:

Conclusions about anthropogenic influences on the atmosphere can only be drawn with respect to the null hypothesis, i.e. one requires a theory of the natural variability, including knowledge of the probabilities of the extremes at various resolutions. At present, the null hypotheses are classical so that they assume there are no long range statistical dependencies and that the probabilities are thin-tailed (i.e. exponential). However we have seen that cascades involve long range dependencies and (typically) have fat tailed (algebraic) distributions in which extreme events occur much more frequently and can persist for much longer than classical theory would allow. Indeed, the problem of statistical “outliers” may generally be a consequence of the failure of highly variable cascade data to fit into relatively homogeneous, regular, classical geostatistical frameworks.

5.3 Perspectives:

In this paper, we have given an overview of a body of work carried out over the last 25 years aiming at a scale by scale understanding of the space-time statistical structure of the atmosphere and its models. The new synthesis we propose would not be possible without technologically driven revolutions in both data quantity and quality as well as in numerical modeling and data processing. Also key for this synthesis are advances in our understanding of nonlinear dynamics (especially cascades, multifractals, and their anisotropic extensions), and in the corresponding data analysis techniques. Although there are many gaps to fill, it is remarkable that a relatively simple picture of the atmosphere as a system of interacting anisotropic cascades seems to be consistent with some of the largest and highest quality satellite, lidar, drop sonde and aircraft campaigns to date collectively measuring passive and active radiances over the long and short wave regimes, as well as in situ wind, temperature, humidity, potential temperature, pressure and other variables. It also holds remarkably well for reanalyses and other numerical models of the atmosphere. It leads to a natural distinction between the weather and climate and successively predicts the transition to the climate at ≈ 2 weeks as a dimensional transition from a weather system (where both long range space and time correlations are important) to a climate system dominated by long range temporal correlations.

At first sight, it may seem surprising that this essentially phenomenological model works so well. However, it is basically a generalization of the most successful laws of classical turbulence, those of Kolmogorov, Obukhov, Corrsin and Bolgiano and it is based three basic features of the governing equations: the scale invariance of the nonlinear terms, the existence of scale by scale conserved fluxes, and the fact that interactions are strongest among structures with similar sizes. The key points of the generalization are to anisotropy so that the finite scale thickness of the atmosphere need not break the horizontal scaling, and to intermittency to account for the extraordinary
variability of the atmosphere explained in this picture as a consequence of the wide range of scales over which the variability builds up.

Although taken individually different parts of the picture presented here can (and must) be criticized. However, given the coherence of all the pieces and the fact that the model is based on fundamental symmetries, we feel it is a promising paradigm for unifying atmospheric dynamics. In any case, some coherent picture is needed to replace the aging and untenable (but still dominant!) 2D isotropic /3D isotropic turbulence model. We are confident that the challenge of improving or replacing this new synthesis will help atmospheric science forward during a period where scientific understanding has all too often been sacrificed for operational expediency.

**Acknowledgements**

We would like to thank John Snow for encouraging us to undertake this mini review. Lovejoy acknowledges NSERC for small discovery grants. Most of the work described here was unfunded.
References


