Scale invariance in climatological temperatures and the local spectral plateau

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ABSTRACT. Instrumental temperatures averaged both locally (over a small region) and hemispherically, as well as paleotemperatures from both ice and ocean cores are analysed statistically on time scales (Δt) of minutes to 9 × 10^4 years.

a) Hemispheric temperatures: For 5 ≤ Δt ≤ 40000 years, characteristic hemispheric temperature changes (ΔT) follow the scale invariant law ΔT ~ 0.07 Δt^-0.5 K which corresponds to a spectrum f^-1, where f is the frequency. For Δt ≥ 8 × 10^4 years, AT is roughly a white noise with standard deviation ± 2.7 K.

b) Local temperatures: For 1 month ≤ Δt ≤ 4000 years, local temperature spectra exhibit a flat « plateau » corresponding to a white noise magnitude ± 0.44 K (ignoring the annual cycle). For Δt ≤ 1 month (the synoptic maximum), and ignoring the diurnal cycle and its harmonics, the spectrum again follows the f^-1 form down to periods of the order of minutes or less.

Key words: climate, scaling, paleotemperatures, spectra, temperature, scale invariance, fractals.


1. INTRODUCTION

Geochemically inferred paleotemperatures have conclusively established the existence of climatological temperature fluctuations at all observed time scales, instrumental records and climatological proxy records of temperature arc used in this paper to show that the energy spectrum of climatological temperature fluctuations with frequency (f) between at least (5 years)^-1 to about (40000 years)^-1 are of the scale invariant form f^-β with β ~ 1.8.

We also spectrally analyse local temperature series. If we ignore the diurnal peak and its harmonics we find that approximately the same value of β is relevant on scales of roughly 1 month (the synoptic maximum) down to periods of the order of minutes or less. This « red noise » spectrum is also very close to the f^-1 spectrum predicted for the spectrum in a turbulent fluid (see section 2.1). The relevance of turbulence fluctuations up to planetary sizes, in the strongly anisotropic atmosphere, is discussed in Schertzer and Lovejoy, 1985b. The following paper is a considerable elaboration of the results discussed in Lovejoy and Schertzer (1983).

The temperature T(t) has fluctuations (ΔT) that are scale invariant, parameter H if they follow the scale invariant relationship:

$$\Delta T(\lambda \Delta t) \sim \lambda^H \Delta T(\Delta t)$$  \hspace{1cm} (1)

where

$$\Delta T(\Delta t) = T(t_1) - T(t_0), \quad \Delta t = t_1 - t_0$$

and

$$\Delta T(\lambda \Delta t) = T(t_2) - T(t_0), \quad t_2 = t_0 + \lambda(t_1 - t_0)$$

for arbitrary scale ratios λ (and Δt). The « \(\sim\) » sign indicates that equality is understood in the sense of probability distributions. The random variables u, v are equal in this sense when P(\(u > q\)) = P(\(v > q\)) for any threshold q (the Pr « indicates « probability »). The parameter 0 ≤ H ≤ 1 is related to the spectral exponent β by the formula β = 2 H + 1 (when the variance is finite). When β > 1, this equation relates fluctuations at small scales (Δt) to those at large time scales (λ Δt). Note that an immediate consequence of this type of scale invariance is that moments of all orders (as well as cross correlations) vary as powers of λ. Another type of scale invariance also of geophysical interest, involves various temporal (or spatial) averages rather than the fluctuations. This type of scale invariance is associated with cascade processes and is now called « multiplicative chaos » — (Kahane, 1985; Schertzer and Lovejoy,
In this case only the extreme fluctuations—not all the fluctuations follow equation (1) (see section 5 for details).

In the spirit of modern physics, we may regard scale invariance (abbreviated « scaling ») as a basic symmetry principle which is expected to hold in the absence of symmetry breaking mechanisms. This approach shows promise in meteorology (Schertzer and Lovejoy, 1984; Lovejoy and Schertzer, 1986a) and is closely related to the approaches used in both renormalisation and fractal modelling. A problem is « symmetric » when it is invariant under a class of transformations (e.g. rotations). In the simplest case of scaling fluctuations in one dimension (\( \Delta \) is a scalar), the fluctuations are « symmetric » because they are invariant under the mathematical operation of magnifying the time scale by \( \lambda \) and rescaling the fluctuations by \( \lambda^2 \). In the more general case of interest, for example when considering fluctuations of temperature in space as well as in time, it is shown that fluctuations may be scale invariant when they are « symmetric » under the action of a more general class of transformations involving not only magnification but also differential stretching and rotation.

If over a certain range in time, fluctuations occur at all scales, with no characteristic period, then over this interval, the fluctuations are scaling. Intuitively, scaling is the simplest way that fluctuations at different scales may be related to each other. An investigation of the limits of scaling is therefore necessary if the basic time scales of a system are to be understood. Strictly speaking, the scale invariance of the fluctuations should be established by directly evaluating the probability distributions of \( \Delta T \) over a wide range of time lags \( \Delta t \). However, this has the disadvantage of requiring very long independent series (see Lovejoy, 1981; Schertzer and Lovejoy, 1986a for meteorological examples), and is postponed until section 5. In sections 2, 3, and 4 we rather investigate various second order statistics—the structure function, spectrum, and rescaled range—all of which show clear evidence of scaling behaviour.

To our knowledge, until recently the only investigations of scaling in series with climatological significance were Hurst (1951), and Mandelbrot and Wallis (1969). Using the range and rescaled range statistics, they were able to substantiate the scaling hypothesis for river discharges, tree rings, waves and precipitation series over various ranges of time scales. The findings of Lovejoy and Schertzer (1983) and the following do not wholly confirm their analyses in that a significant difference is found between locally and globally averaged temperature series with the former displaying a long « spectral plateau » apparently absent in the latter. Both the onset of the plateau at \( \Delta t \sim 1 \) month, and its end at \( \Delta t \sim 450 \) years mark breaks in the scaling of local (but not necessarily global) temperatures. Another relevant reference is Berger and Piestaux (1984) and Piestaux (1984), where spectral analysis of ocean core paleotemperatures show that the spectral form \( f^{-\beta} \) does not extend to frequencies much less than about \( (10^6 \text{ years})^{-1} \).

The scaling spectrum \( f^{-\beta} \) has an « excess » of energy at low frequencies and thus (in the sense of Gilman et al., 1963) is a « red noise ». In climatological temperatures, the importance of this excess has long been recognized (e.g. Lamb et al., 1966). It has also been known for some time that the « background » of empirical spectra often have broadly \( f^{-\beta} \) forms (e.g. Hays et al., 1976). Furthermore, several authors (e.g. Hasselmann, 1976; Bhattacharya et al., 1982; Le Treut and Ghil, 1983) have proposed theoretical models with \( f^{-\beta} \) background spectra with \( \beta = 2 \) (the latter model can also allow for \( \beta < 2 \) — Ghil, private communication). The present paper builds on these results in two directions. First, we show that the scaling extends much further than previously believed, and second, that the exponent \( \beta \) is significantly less than 2 \((\beta = 1.82 \pm 0.04)\) and is very close to that determined for local temperature fluctuations on a scale of minutes (and perhaps seconds) up to several weeks. This raises the intriguing possibility that the scaling of the global mean temperature—if it could be measured at these scales—may continue down to periods as short as seconds. In that case, for periods less that ~40000 years, climatological and meteorological temperatures would differ fundamentally only in the time and space scales over which they are averaged.

2. ANALYSIS OF INSTRUMENTALLY DETERMINED TEMPERATURE SERIES

2.1. Local temperatures

Temperature is of necessity a spatially averaged quantity. As temperatures are averaged over progressively larger regions, eventually covering the entire globe, the amplitude of \( \Delta T \) for a fixed \( \Delta t \) will decrease because averaging « smooths out » any large local variations. It is therefore important to carefully distinguish the globally averaged fluctuations (referred to as « global »), from these averaged over only a small part of the earth's surface (referred to as « local »).

Figure 1a shows the energy spectrum of temperature fluctuations from the climatological recording station in Macon, France from 1951-1980. At the high frequency end (2 days) \(^{-1}\) to about (1 month) \(^{-1}\), the spectrum follows the straight line on a log-log plot corresponding to \( E(f) \propto f^{-1.8} \). Over the range (2 days) \(^{-1}\) to (6 h), the spectrum is dominated by the diurnal peak and various subharmonics. At higher frequencies, figure 1b shows that the \( f^{-1.8} \) form continues down to at least (10 mm) \(^{-1}\). Other evidence for \( \beta \sim 1.8 \) in this high frequency region can be obtained from spatial (wavenumber) spectra by the commonly used technique of transforming between spatial and temporal statistics with an appropriate velocity factor (Taylor's hypothesis of « frozen turbulence »). If this transformation is valid it implies that the values of \( \beta \) for both frequency and wavenumber spectra should be the same. Relevant empirical results on temperature wavenumber spectra are Tatarski (1956) \((k^{-1.83})\) and Deschamps et al., (1981) \((k^{-1.81.91})\). These spectra are very close to the \( f^{-1.8} \) form predicted by Corsoin (1951) and Obukhov (1949) for isotropic, homogeneous turbulence (for a
contemporary discussion, see Herring et al. (1982) and for extensions to the anisotropic turbulence in the atmosphere, see Schertzer and Lovejoy, 1985a). An interesting possibility is that this « background » follows $f^{-1.8}$ up to frequencies of $(1 \text{ s})^{-1}$.

The high frequency scaling regime clearly breaks down for $f \lesssim (1 \text{ month})^{-1}$ and the spectrum becomes fairly flat — the « spectral plateau » referred to earlier. This plateau, « white noise » spectral region shows that $E(f)$ is almost constant at these scales. If we had plotted $f E(f) \log f$ (rather than $\log E(f) \log f$) as was done in Kolesnikov and Monin (1965), we would have obtained a peak instead of a smooth transition at $f \sim (20 \text{ days})^{-1}$ — the « synoptic maximum ». These authors estimated the period of the maximum to be between 1 and 3 weeks. It is generally agreed that this is the minimum time scale for fluctuations of planetary size.

With the exception of the sharp peak at $f \sim (1 \text{ year})^{-1}$, the spectral plateau continues down below the lowest frequencies available at Macon (30 years)$^{-1}$ to at least (300 years)$^{-1}$, which is the frequency associated with the longest instrumental temperature series (see Mason’s (1976) analysis of Manley’s (1974) central England data).

A way of exhibiting the spectral plateau which will be useful in later discussions is to consider the structure function $[S(\Delta t)]$. This is defined as the square root of the mean square fluctuation:

$$S(\Delta t) = \langle [\Delta T(\Delta t)]^2 \rangle^{1/2}.$$
$S(\Delta t)$ gives us direct information about the likely magnitude of temperature differences for time intervals $\Delta t$. Figure 2 shows the $S(\Delta t)$ function for the Manley series, showing that over this period, $S(\Delta t)$ is almost constant; the amplitude 0.08 K corresponds to a white noise of amplitude 0.04 K. This is the spectrum function counterpart of the local spectral plateau.

At the very low frequencies corresponding to the period of the interglacials, we know that $S(\Delta t)$ must be significantly larger. For example, Emiliani and Shackleton (1974) estimate that there were strong oscillations in the temperature of the entire earth in the last 4-6 x 10^6 years of amplitude $\pm 3$ K. Thus, during a half-period $(\Delta t)$ of 3-5 x $10^6$ years, $S(\Delta t)$ has the value 3 K (there is some agreement that this amplitude is between 2 and 4 K). Therefore, at these scales, the entire temperature of the earth varies considerably more than that of local regions at scales of hundreds of years. A convenient way of visualizing this is to plot the «interglacial window» (i.e. the region of the $S(\Delta t)$ plane where the global $S(\Delta t)$ function must cross) as shown in the square box in figure 2. At some point, apparently for $\Delta t > 300$ years, the spectral plateau must end and $S(\Delta t)$ will rise and pass through the «window». Below, we argue that the end of the plateau occurs at $\Delta t \sim 450$ years.

2.2. Hemispheric temperatures

In the previous subsection, we have argued that over short-time scales, global average temperatures have considerably smaller amplitude variations than local ones whereas at long enough time scales, the amplitude of the local fluctuations are likely to be primarily due to the overall temperature variation of the entire globe. This may be expressed by saying that the global variations eventually become large enough to dominate the local ones. To illustrate this idea, we analyse the Budyko (1969) and Jones et al. (1982) northern hemisphere temperature series for the last century (unfortunately, data from the southern hemisphere is too sparse to warrant the estimation of global averages). Figure 2 shows the $S(\Delta t)$ function for these two series. The Budyko (1969) series is not shown for $\Delta t < 5$ years since a moving 5 year average was used. Note that although the Jones et al. (1982) «optimum grid» data extend down to periods of 1 month (see fig. 3), the average annual cycle was removed.

Estimating the mean hemispheric temperature is not a simple task because measuring stations are distributed highly inhomogeneously over the surface of the earth. Lovejoy et al. 1986 show that in geophysical measuring networks the inhomogeneities can occur over a wide range of scales, reflecting the fact that the fractal dimension of the stations (considered as a set of points on a sphere) is less than 2 (the value corresponding to a uniform distribution). Interpolating onto a higher dimensional space (e.g. a uniform grid) is therefore a difficult task and generally involves biases in the statistical properties of the measured fields (see Lovejoy et al., 1986; Ludy, 1985; Ludy et al., 1986; Lovejoy and Schertzer, 1986a for more details). The difference between the Budyko and Jones et al. curves in figure 2 is due both to the different estimation techniques and data bases used. Budyko subjectively selected «representative» measurements from data rich regions in an attempt to obtain a uniform sampling of temperatures. Jones et al. (1982) chose a particular objective technique which linearly interpolates temperatures on a regular grid, before the averages are taken.

Although the sampling methods differ, we would not expect different averaging or sampling methods to alter either the overall form of the $S(\Delta t)$ function or of the spectrum. In spite of these differences, figure 2 shows that the ratio of the $S(\Delta t)$ functions is roughly a constant equal to 1.20. This important point requires some discussion. The situation would be quite different if the two series differed primarily by a random measurement error. In that case, the difference between the variance [$S^2(\Delta t)$] would be constant (equal to the sum of the error variances), and would imply that log-log plots of the Budyko and Jones et al. curves would approach each other for large $\Delta t$. In particular, if we distinguish the Budyko and Jones et al. $S(\Delta t)$ functions by the subscripts $B$, $J$, respectively, then the plot of $log(S_B - S_J)$ vs. log $\Delta t$ will have a slope $\sim 0.4$ if the fluctuations have a constant ratio, but will have slope 0 if the difference between the two is constant. Such an analysis does indeed support the constant ratio hypothesis.

The constant ratio may be explained if we consider that neither series has sufficient sampling density to measure the true northern hemisphere temperature only to allow for more or less spatially averaged estimates. If, over the scales considered (here $\Delta t > 2$ years), the sampling and averaging do not introduce a characteristic time then they will not change the shape of the $S(\Delta t)$ function, hence, differences will be a constant ratio.

We therefore interpret the fact that $S_B < S_J$ to indicate that the Budyko series has a higher degree of spatial averaging. It is therefore probably closer to the true $S(\Delta t)$ function, and will be used below. The most important point to note about these empirical $S(\Delta t)$ functions is that unlike the Mantle (local) temperature $S(\Delta t)$ function, when $\Delta t$ is sufficiently large, these
hemispheric $S(\Delta t)$ follow the scaling function:

$$S(\Delta t) = 0.077 \Delta t^H, \quad H \sim 0.4 \quad \text{(Budyko, 1969)}$$

$$= 0.062 \Delta t^H, \quad H \sim 0.4 \quad \text{(Jones et al., 1982)}$$

the former is indicated as the straight line in figure 2. Note that the exponent $H \sim 0.4$ corresponds to a spectral exponent of $\beta = 2H + 1 = 1.8$. Figure 3 shows a log-log plot of the spectrum which confirms that for low frequencies $\beta \sim 1.8$. At frequencies higher than (5 years)$^{-1}$, there is apparently a flat plateau. The exact point of transition is somewhat difficult to judge: the $S(\Delta t)$ curve suggests it occurs at $\Delta t \sim 8$ years, while the spectrum, $\Delta t \sim 3$ years. We choose the geometric mean 5 years although this is obviously a crude estimate. It is not clear whether, as in the local temperature spectrum, this is true spectral plateau, or whether it is an artefact resulting from the difficulty of accurately measuring such small temperature fluctuations. To put this problem in perspective, the variance associated with this high frequency noise is (0.21 K)$^2$. In contrast, if we assume that the scaling (straight-line) in figure 2 continues down to 1 month, then a measurement precision of 0.092 (1/12)$^{0.5} = 0.034$ K would be required to clearly show it. For comparison, Jones et al. (1982) cite a difference of between 0.16 K and -0.08 K for the Budyko (1969) series, and that of another series compiled by Vinnikov et al. (1980). Existing series may therefore not be adequate for resolving this problem.

If we extrapolate the low frequency scaling using the equation $S(\Delta t) \sim 0.077 \Delta t^{0.4}$ for $\Delta t$ up to 4000 years, we find that it passes right through the «window» defined by the interglacials. Indeed, if we assume that between $\Delta t = 5$ years (where $S(\Delta t) = 0.15$ K for the Budyko series) and the limits of the interglacial «window» in figure 2, that there is no fundamental time scale (i.e. $S(\Delta t)$ is of the form $\Delta t^{H}$) then $H$ is constrained to lie in the range 0.36 $\leq H \leq 0.4$. Since the empirical $H$ lies in this interval, a simple extrapolation from the current northern hemisphere $S(\Delta t)$ function therefore is consistent with the frequency and magnitude of the interglacials. In the next section, we show that this behaviour, including the value of $H$, is also fully consistent with many of the existing paleotemperature series.

Note that at $\Delta t = 5$ years, the average northern hemisphere temperature fluctuations ($\pm 0.074$ K) are 6.0 times less than the Manley (local) fluctuations of $\pm 0.44$ K. The local temperature fluctuations therefore dominate the hemispheric ones until 0.077 $\Delta t^{0.4} > 0.88$ or until $\Delta t \geq 450$ years - a result which is consistent with both the Manley series, with the Bryson and Katzbach (1974) estimated spectra (from botanical records) and, with the spectral inflection point at 400 years shown in Mitchell (1976).

3. PALEOTEMPERATURE SERIES ANALYSIS

3.1. Ice cores

The investigation of the scaling hypothesis at scales of the order of centuries or longer, requires the use of paleotemperature (proxy) data. Probably the most reliable for this purpose are the oxygen isotope ratios ($^{18}O/^{16}O$) determined from Arctic and Antarctic ice cores. Ice cores have a deficit of $^{18}O$ because the lighter ($^{18}O$) evaporates more readily from the oceans than the heavier $^{18}O$. This deficit is highly correlated with the temperature at the time of evaporation and deposition (see Duplessy (1978) for a discussion). If this is true, then we expect $^{18}O/^{16}O$ ratios to provide an estimate of
local temperature because the local environment over the ice cap is expected to strongly influence these processes. It would seem reasonable to assume that the O\textsuperscript{18}/O\textsuperscript{16} ratios in these cores are proxies for the mean temperature of a region greater than Manley's central England area, but nonetheless, considerably smaller than the entire northern hemisphere. We therefore expect that at sufficiently short time scales (where the local fluctuations dominate the hemispheric ones), the S(\Delta t) function will exhibit a flat plateau. However, due to the larger effective area of spatial averaging, it should have a somewhat smaller value of S(\Delta t) than in the corresponding plateau region for central England. Again, for \Delta t \gtrsim 450 years, the hemispheric fluctuations are likely to dominate yielding S(\Delta t) \sim 0.077 \Delta t^{-0.4} behaviour.

Figure 4 shows the $S(\Delta t)$ functions for both arctic and antarctic series, taken from Johnson et al., (1972). We used their dating and the linear temperature calibration constant was adjusted so that these $S(\Delta t)$ functions would lie on the lines $S(\Delta t) = 0.077 \Delta t^{-0.4}$ on the log-log plot shown in figure 4 (i.e. the curves were moved up and down on the log-log plot until the straight-line sections matched this function). Note that when this is done, the $S(\Delta t)$ curves pass through the interglacial «window», reaching a maximum at about 6-8 \times 10^4 years (a behaviour characteristic of $S(\Delta t)$ functions of oscillatory series). For the largest $\Delta t$, the $S(\Delta t)$ function corresponds to the plateaux temperature fluctuations of \pm 2.7 K. The $S(\Delta t)$ values were obtained from both high resolution (every \sim 100 years for 2 \times 10^4 years) as well as low resolution (every \sim 700 years for the last 1.2 \times 10^5 and 9 \times 10^4 years for arctic and antarctic series respectively).

In this analysis we do not wish to minimize the calibration problems inherent in the use of paleotemperatures. For example, we have ignored the possibility of a small non-linear correction to the O\textsuperscript{18}/O\textsuperscript{16} temperature relationship (see Duplessy, 1978). We have also assumed that the sample dating was correct although as long as this contained only a linear error (corresponding to a left-right shift of the $S(\Delta t)$ points in figure 4), it could only be distinguished from a linear temperature calibration error (an up-down displacement) at the curved (small and large $\Delta t$) ends of the $S(\Delta t)$ function. Although non-linear calibration problems undoubtedly influence the shape of the $S(\Delta t)$ function, as long as they are of second order they should not alter the scaling exponent $H$ found in the scaling regime.

It is of considerable interest to clarify the nature of the maximum in $S(\Delta t)$ at scales of $\Delta t \gtrsim 40000$ years, which are only poorly resolved by these series. In order to do this, we turn to the examination of ocean cores, which make it possible to investigate $S(\Delta t)$ up to $\Delta t \sim 9 \times 10^6$ years.

3.2. Ocean cores

The interpretation of deep-sea cores is not as straightforward as that of ice cores. For example, O\textsuperscript{18}/O\textsuperscript{16} ratios may be obtained from the calcium carbonate of fossil plankton. In this case, the ratios depend not only on the temperature of the surface layer where the plankton once lived; but also on the O\textsuperscript{18}/O\textsuperscript{16} ratio in their surrounding water, which in turn depends on the volume of water in the ice caps. It is now generally agreed that the latter is the dominant effect (see Duplessy, 1978), implying that the ocean core ratios are best interpreted in terms of the volume of water in the ice caps. However, on time scales of \sim 10^5 to 10^6 years, fluctuations in the size of the ice caps are correlated with the mean global temperature (although probably with an associated time lag — Bhattacharya et al., 1982; Le Treut and Glai, 1983). Other complications include bioturbation effects in the sediments from which the cores are taken (e.g. Goreau, 1980). Indeed, Dafies et al. (1984) has even suggested that at least for scales $\Delta t \lesssim 10^5$ years, the latter might in fact be a scaling, first order effect and could therefore modify $H$ for shorter time scales. In spite of all these complications, ocean cores may still be expected to permit us to examine the extent of the scaling regime and at least to get a rough idea of the fluctuations at very long periods. Let us stress that the basic conclusions of this paper (i.e. the existence of a spectral plateau, and global scaling up to \sim 40000 years) are based entirely on the instrumental and ice core series discussed earlier. The ocean cores are only used to get a feel for the temperature variations over very long time periods — especially for $\Delta t \gtrsim 10^5$ years.

In figure 4, we have plotted data analysed from Shackleton and Opdyke (1973). The $S(\Delta t)$ curve obtained has a scaling behaviour, with $H \sim 0.4$ for roughly an order of magnitude in $\Delta t$ up to $\Delta t \sim 4 \times 10^6$ years, after which it is fairly constant. The linear calibration was performed by aligning the $H \sim 0.4$ part to the ice core data using the dating specified in Shackleton and Opdyke (1973). This curve fits the ice data fairly well except for the region near the peak (where the latter has less statistical significance anyhow). If this peak is real, then it is an indication of oscillatory behaviour of $\Delta T$ on a scale of $2 \times 4 \times 10^8 \approx 8 \times 10^6$ years (the factor 2 is necessary because $4 \times 10^6$ corresponds to a half-period).

Finally, it should be noted that the recent spectral analysis of ocean cores in Berger and Pestiaux, 1984; Pestiaux, 1984 tends to confirm that the $H \sim 0.4$ behaviour (f^{-1.8} spectrum) cannot continue to frequencies much below (10^4 years)^{-1}.
4. THE ACCURACY OF THE SCALING EXPONENT $H$

It is important to establish the accuracy of our estimate of $H$ since the amplitude of $\Delta T$ for $\Delta t \approx 40000$ years is very sensitive to its exact value (as indicated in figure 2). For example, $H = 1/2$ corresponding to the $\beta = 2$ «background spectrum» yields a temperature change of $13.4 = \pm 0.7$ K which is probably too large to be compatible with other estimates of the magnitude of $\Delta T$, and $H = 1/3$ (the value corresponding to the $f^{-5/3}$ Kolmogorov spectrum) yields $\Delta T^2 = 3.0 = \pm 1.5$ K, which is probably too small.

In order to accurately measure the value of $H$, a robust measure of long-run dependence known as R/S analysis was used (see Mandelbrot, 1972). This statistic is robust, because it is insensitive to large fluctuations around the long-term trend. It has the disadvantage that it is biased for short time series. When it was applied to the Arctic and Antarctic ice core series in the range $4 \times 10^3 \lesssim \Delta t \lesssim 4 \times 10^4$ years, this method yielded $H = 0.420$, and $0.404$ respectively (with correlation coefficient of 0.955, 0.985 respectively). We therefore estimate $H = 0.4 \pm 0.02$. However, a formula more precise than $S(\Delta t) \sim 0.077 \Delta t^{0.4}$ is not warranted because of the uncertainties in the values of $S(1)$. Note that care has been taken to apply R/S analysis only over scales not affected by the spectral plateau.

5. THE SCALING OF EXTREME FLUCTUATIONS

In the preceding sections we have examined various second order statistics of the temperature fluctuations. If equation (1) holds exactly then the probability distribution of the fluctuations scale, and hence not only the second order statistics but those of all other orders also scale. However, if scale invariance is associated with averages and not fluctuations (as it does in multiplicative chaos), equation (1) still holds but now only for the extremes (large $\Delta t$) (e.g. Schertzer and Lovejoy, 1985b). In any case the scaling of the extremes is important in its own right because it is a fundamental characteristic of the climate's intermittency.

To investigate the scaling of the probability distributions determine $\text{Pr}(\Delta T(\Delta t) > \Delta T)$ for various time lags $\Delta t$. This is the probability that a random fluctuation $\Delta T(\Delta t)$ exceeds a fixed $\Delta T$ (in the following the argument $\Delta t$ will be understood implicitly). If equation (1) holds, then the distributions are scaling with parameter $H$. Increasing time scales by a factor $\lambda$ ($\Delta t \rightarrow \lambda \Delta t$) increases fluctuations by $\lambda^H$ hence on a plot of $\log(\text{Pr}(\Delta T(\Delta t) > \Delta T))$ against $\log \Delta T$ this yields a linear shift of $H \log \lambda$. If only the extremes scale, then the shift is constant only for the probability tail (large $\Delta t$). Figure 5 shows the probability distributions when $\lambda = 4, 16, 64$ for both the Arctic ice core and for the Jones et al., 1982 northern hemisphere data. Although the distributions are quite «noisy» the tails if not the rest of the distributions are fairly similar in shape for the different lags. Furthermore for $\Delta t$ within the previously defined scaling regimes (the entire range in figure 5a, and the 16 and 64 year curves in figure 5b), the shift is reasonably close to 0.4 log 4 as expected for $H = 0.4, \lambda = 4$.

Concentrating on the extremes (large $\Delta t$), figure 5 displays another important feature — the probability «tail» is nearly straight. For reference straight lines of slope $-\alpha$ separated by $H \log 4$ are shown with $H = 0.4, \alpha = 5$.

Empirically almost exactly this type of hyperbolic probability tail (i.e. $\text{Pr}(\Delta T > \Delta T) \Delta T^{-\alpha}$ for large $\Delta T$), has been found in local and regional daily temperatures (Lovejoy and Schertzer, 1985) as well as in a variety of other meteorological fields such as the rain, wind and potential temperatures (see e.g. Lovejoy, 1981; Lovejoy and Mandelbrot, 1985; Schertzer and Lovejoy, 1985a; Lovejoy and Schertzer, 1985). Theoretically, hyperbolic probability tails may be expected to occur as a result of cascade processes (Mandelbrot, 1974; Schertzer and Lovejoy, 1983, 1985b).

Figure 5a
A log-log plot of $\text{Pr}(\Delta T > \Delta T)$ which is the probability of a random temperature fluctuation $\Delta T$ exceeding a fixed threshold $\Delta T$. The data used are from the arctic series with the linear temperature calibration obtained from figure 4. The curve to the far left is for fluctuations over intervals (\Delta t) of 350 years. The remaining curves, from left to right are obtained by increasing $\Delta t$, by factors of 4 (i.e. 1400, 5600, 22400 years respectively). The straight lines indicate the functional $\text{Pr}(\Delta T > \Delta T) \sim (\Delta T/\Delta T)^{\alpha}$ with $\alpha = 5$ and the amplitude of the fluctuations $\Delta T$ varying as $\Delta T \sim \Delta T^{\alpha}$ as required by scaling; with $H = 0.4$.

Figure 5b
The same as figure 5a except for the northern hemisphere annual average (Jones et al., 1982) data. The curves from left to right are for $\Delta t = 1, 4, 16, 64$ years. The straight lines are for the same values of $H, \alpha$, as in figure 5a. This figure confirms that the scaling regime is obtained only for $\Delta t > 4$ years.
6. INTERPRETATION

6.1. Global scaling

If climatological temperature fluctuations are to be eventually explained by the action of invariant physical laws, then it is important to consider the question of the stationarity of the temperature. The preceding analysis shows that we must make a basic distinction between the true stationarity of the process, and its apparent non-stationarity for periods of less than 40000 years. True stationarity means that the process is invariant with respect to translations along the time axis — hence that the ensemble mean temperature is independent of time. In the present case, temperature averages over periods of less than 40000 years will vary due to the scaling behavior. This apparent non-stationarity is responsible for the difficulty in defining climatological temperatures for shorter periods. However, over longer periods, the break in the scaling does permit average temperatures to be well defined. If, we examine temperature increments (i.e., the series defined by the first differences of T), the situation is much simpler, because, Lovejoy and Schertzer (1983) have verified that the latter are apparently stationary over the entire range of 5 to 9 x 10^6 years.

With the above discussion in mind, it is possible to imagine many different mechanisms to account for the empirical S(Δt) function. For example, dynamical system theory might lead us to interpret the change in slope at Δt ~ 40000 years as marking the boundary (characterized by the magnitude ± 2.7 K) of a regime dominated the cascade of temperature variance flux to shorter and shorter time scales via non-linear interactions. Another interpretation, not necessarily in contradiction to the first, is that in the scaling regime, the temperature changes by the accumulation of random fluctuations. If these fluctuations were independent, then the temperature would vary in the same way as the co-ordinate of a Brownian particle: we would have obtained (as in Hasselman, 1976) H = 1/2. The fact that H < 1/2 indicates that fluctuations have a tendency to cancel, even over very long time periods. Physically, this could result from the existence of negative feedback between the various parts of the climate system operating over a wide range of time scales. Note that the background spectrum while being typical of cascade processes, does not itself permit us to draw conclusions about the forcing mechanism — whether astronomical or otherwise (for the former, see the review Berger, 1980).

6.2. The local spectral plateau

For local temperatures, we have argued that there is spectral plateau extending down to f ~ (450 years)^-1. However, we have seen that for hemispherically averaged temperatures, the plateau, if it is real, ends at f (3 years)^-1. We therefore expect the plateau to diminish in size as the area over which the temperature is averaged is increased. A detailed study of this phenomenon would be illuminating.

7. CONCLUSIONS

We may summarize our results as follows:

a) Local temperatures have a scaling regime extending from at least several minutes to month with β ~ 1.8 (H ~ 0.4).

b) For scales of 1 month to 450 years, the local temperatures have a flat spectrum which we call the local spectral plateau, of amplitude ± 0.44 K. After 450 years, they are dominated by hemispheric variations, and apparently follow the scaling law S(Δt) ~ 0.077 Δt^H with H ~ 0.4 up to Δt ~ 4 x 10^4 years.

c) For Δt ≥ 5 years, the hemispheric fluctuations follow the scaling law 0.077 Δt^H up to Δt ~ 4 x 10^4 years. For Δt ≥ 5 years, these fluctuations are apparently characterized by a spectral plateau, although this could be an artifact, if the climate network has an accuracy lower than ± 0.075 K. In this case, the scaling may continue to significantly higher frequencies.

d) For Δt ≥ 4 x 10^4 years, the temperature variations are roughly stationary, yielding oscillatory behavior with period 2 x 4 x 10^4 = 8 x 10^4 years with an amplitude ± 2.7 K.

Although we believe that the various data sources are fairly representative of the climatological phenomena discussed and that the above conclusions are valid, it is obvious that considerable additional work will be necessary, including the examination of other series.

If the above analysis is correct, then it may be possible to calibrate paleotemperatures by fitting the empirical S(Δt) curve to that shown in figure 2. This method would work if the required calibration in either temperature or age is approximately linear.

Finally, it is interesting to note that the fundamental difference between meteorological and climatological temperatures is determined by the manner in which the high and low frequency parts of the spectrum join and how they vary when averaged over various time and space scales. In this regard, the very similar form (f^-1.8) of the climatological and turbulent spectra is very suggestive.

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