Low frequency weather and the emergence of the Climate

S. Lovejoy¹, D. Schertzer²,³
¹ Physics, McGill University, 3600 University St., Montreal, Que. H3A 2T8, Canada
² LEESU, Ecole des Ponts ParisTech, Université Paris Est, France
³ Météo France, 1 Quai Branly, Paris 75005, France

Abstract

We survey atmospheric variability from weather scales up to several 100 kyrs. We focus on scales longer than the critical $\tau_w \approx 5$-20 day scale corresponding to a drastic transition from spectra with high to low spectral exponents. Using anisotropic, intermittent extensions of classical turbulence theory, we argue that $\tau_w$ is the lifetime of planetary sized structures. At $\tau_w$ there is a dimensional transition at longer times, the spatial degrees of freedom are rapidly quenched, leading to a scaling “low frequency weather” regime extending out to $\tau_c \approx 10$-100 yrs. The statistical behaviour of both the weather and low frequency weather regime are well reproduced by turbulence-based stochastic models and by control runs of traditional GCM’s, i.e. without the introduction of new internal mechanisms or new external forcings, hence it is still fundamentally “weather”. Whereas the usual (high frequency) weather has a fluctuation exponent $H > 0$ implying that fluctuations increase with scale, in contrast, a key characteristic of low frequency weather is that $H < 0$ so that fluctuations decrease instead. Therefore, it appears “stable” and averages over this regime (i.e. up to $\tau_c$) define climate states. However, at scales beyond $\tau_c$ - whatever the exact causes - we find a new scaling regime with $H > 0$ i.e. where fluctuations again increase with scale, climate states thus appear unstable, this regime is thus associated with our notion of climate change.

We use spectral and difference and Haar structure function analyses of reanalyses, multiproxies and paleotemperatures.
1. Introduction

1.1. What is the climate?

Notwithstanding the explosive growth of climate science over the last twenty years, there is still no clear universally accepted definition of what the climate is – or what is almost the same thing – what the difference is between the weather and the climate. The core idea shared by most climate definitions is famously encapsulated in the dictum: “The climate is what you expect, the weather is what you get” (see (Lorenz, 1995) for a discussion). In more scientific language “Climate in a narrow sense is usually defined as the "average weather," or more rigorously, as the statistical description in terms of the mean and variability of relevant quantities over a period of time ranging from months to thousands or millions of years (Intergovernmental Panel on Climate Change. Appendix I: Glossary. Retrieved on 2007-06-01).

An immediate problem with these definitions is that they fundamentally depend on subjectively defined averaging scales. While the World Meteorological Organization defines climate as 30 year or longer variability, a period of two weeks to a month is commonly used to distinguish weather from climate so that even with these essentially arbitrary periods, there is still a range of about a factor 1000 in scale (30 years /2 weeks) that is up in the air. This fuzzy distinction is also reflected in numerical climate modelling since Global Climate Models are fundamentally the same as weather models but at lower resolutions, with a different assortment of subgrid parametrisations and they are coupled to ocean models - and increasingly – to cryosphere, carbon cycle and land use models. Consequently, whether we define the climate as the long-term weather statistics, or in terms of the long-term interactions of components of the “climate system”, we still need an objective way to distinguish it from the weather. These problems are compounded when we attempt to objectively define climate change.

However, there is yet another problem with this and allied climate definitions: they imply that climate dynamics are nothing new: that they are simply weather dynamics at long time scales. This seems naïve since we know from physics that when processes repeat over wide enough ranges of space or time scale, qualitatively new climate laws should emerge from the higher frequency weather laws. These “emergent” laws could simply be the consequences of long range statistical correlations in the weather physics in conjunction with qualitatively new climate processes – due to either internal dynamics or to external forcings - their nonlinear synergy giving rise to emergent laws of climate dynamics.

Since the atmosphere is a nonlinear dynamical system with interactions and variability occurring over huge ranges of space and time scales (millimetres to planet scales, milliseconds to billions of years, ratios \( \approx 10^{10} \) and \( \approx 10^{20} \) respectively), the natural approach is to consider it as a hierarchy of processes each with wide range scaling, i.e. each with nonlinear mechanisms that repeat scale after scale over potentially wide ranges. Following (Lovejoy and Schertzer, 1986), Schmitt et al., 1995, Pelletier, 1998, (Koscielny-Bunde et al., 1998), (Talkner and Weber, 2000), (Blender and Fraedrich, 2003), (Ashkenazy et al., 2003; Huybers and Curry, 2006b; Rybski et al., 2008) this approach is increasingly superseding earlier approaches that postulated more or less white noise backgrounds with a large number of spectral “spikes” corresponding to many different quasi-periodic processes. This includes the slightly more sophisticated variants
e.g. (Mitchell, 1976) which retain the spikes but replace the white noise with a hierarchy of Ornstein-Uhlenbeck processes (white noises and their integrals); in the spectrum, “spikes” and “shelves”; see also (Fraedrich et al., 2009) for a hybrid which includes a single (short) scaling regime.

Fig. 1a: A modern composite based only on two sources: the Summit GRIP core (Greenland paleotemperatures) and the Twentieth Century (20CR) reanalyses at the same latitude (75°N). All spectra have been averaged over logarithmically spaced bins, 10 per order of magnitude and the 20CR spectra have been averaged over all 180 longitude, 2°x2° elements, frequency units: (yrs)^{-1}. The light green is the mean of the GRIP 5.2 resolution data for last 90 kyr and the (lowest) frequency blue is from the lower (55 cm) resolution GRIP core interpolated to 200 yr resolution and going back 240 kyr. The solid reference lines have absolute slopes $\beta_{lw} = 0.2$, and $\beta_{c} = 1.4$ and $\beta_{w} = 2$ as indicated. The red arrows at the bottom (and upper right) indicate the basic qualitatively different scaling regimes. Reproduced from (Lovejoy and Schertzer, 2011).

Over the past 25 years, scaling approaches have also been frequently applied to the atmosphere, mostly at smaller scales but in the last five years increasingly to global scales. This has given rise to a new scaling synthesis covering the entire gamut of meteorological scales from milliseconds to beyond the $\approx 10$ day period which is the typical lifetime of planetary structures i.e. the weather regime. In the review (Lovejoy and Schertzer, 2010), it was concluded that the theory and data were consistent with wide range but anisotropic spatial scaling, and that the lifetime of planetary sized structures, provides the natural scale at which to distinguish weather and a qualitatively different lower frequency regime. Fig. 1a shows a recent composite indicating the three basic regimes covering the range of time scales from $\approx 100$ kyr down to weather scales. The label “weather” for the high frequency regime seems clearly justified, and requires no further comment. Similarly the lowest frequencies correspond to our usual ideas of
multidecadal, multicentennial, multimillennial variability as “climate”. However labelling the
intermediate region “low frequency weather” - rather than say “high frequency climate” – needs
some justification. The point is perhaps made more clearly with the help of fig. 1b which shows
a blowup of fig. 1a with both global and locally averaged instrumentally based spectra as well
the spectrum of the output of the stochastic Fractionally Integrated Flux (FIF) model (Schertzer
and Lovejoy, 1987) as well as the spectrum of the output of a standard GCM “control run” i.e.
without special anthropogenic, solar, orbital or other climate forcings. This regime is therefore
no more than “low frequency weather”, it contains no new internal dynamical elements, nor any
new forcing mechanism. As we discuss below, whereas the spectra from data (especially when
globally averaged) begin to rise for frequencies below \( \approx (10 \text{ yrs})^{-1} \), both the FIF and GCM
control runs maintain their gently sloping “plateau” – like behaviours out to at least \((500 \text{ yrs})^{-1} \)
(note that we shall see that the “plateau” is not perfectly flat but its logarithmic slope is small,
typically in the range -0.2 to -0.6). Similar conclusions for the control runs of other GCMs at
even lower frequencies were found by (Blender et al., 2006) and (Rybski et al., 2008) so that it
seems that in the absence of external climate forcings, the GCM’s reproduce the low frequency
weather regime but not the lower frequency spectrally rising regime which requires some new
climate ingredient. The aim of this paper is to understand the natural variability so that the
important question of whether or not GCM’s with realistic forcings might be able to reproduce
the low frequency climate regime is outside our present scope. Certainly existing studies of the
scaling of forced GCM’s (Vyushin et al., 2004), (Blender et al., 2006) and (Rybski et al.,
2008), have consistently reported unique low frequency weather exponents and this, even at the
lowest simulated frequencies.

Fig. 1b: A comparison of the spectra of temperature fluctuations from the the GRIP Greenland
Summit core, last 90 kyr, 5.2 yr resolution, (green, upper left), monthly 20CR reanalysis (blue),
global (bottom) and 2° resolution (top, at 75°N) and a 500 yr control run of the monthly Institut
Pierre Simon Laplace, (IPSL) GCM (brown) used in the 4th IPCC report, at the corresponding
resolutions (frequency units: \((\text{yrs})^{-1}\)). The dashed lines are the detrended daily, grid scale data
(75°N 20CR dashed blue) and the cascade based Fractionally Integrated Flux simulation (dashed red) both adjusted vertically to coincide with the analyses of the other, monthly scale data. Reference lines with slopes $\beta_c = 1.4$, $\beta_{lw} = 0.2$, $\beta_w = 1.8$ are shown. Notice that the IPSL control run – which lacks external climate forcing and is therefore simply low frequency weather as well as the low frequency extension of the cascade based FIF model – continue to have shallow spectral slopes out to their low frequency limits whereas the globally averaged 20CR and paleo spectra follow $\beta_c \approx 1.4$ to roughly their low frequency limits. Hence the plateau is best considered “low frequency weather”: the true climate regime has a much steeper spectrum determined either by new low frequency internal interactions or by the low frequency climate “fo forcing” (solar, orbital, volcanic, anthropogenic or other), fig. 1a shows that the $\beta_c = 1.4$ regime continues to $\approx (100$ kys)$^{-1}$. 

2. Temporal scaling, weather, low frequency weather, and the climate

2.1 Discussion

Although spatial scaling is fundamental for weather processes, time scales much greater than $\tau_w \approx 10$ the spatial degrees of freedom essentially collapse (via a “dimensional transition”), so that we focus on temporal variability (section 2.5). In order to simplify things as much as possible in section 2, we will only use spectra.

Consider a random field $f(t)$ where $t$ is time. Its “spectral density” $E(\omega)$ is the average total contribution to the variance of the process due to structures with frequency between $\omega$ and $\omega + d\omega$ (i.e. due to structures of duration $\tau = 2\pi/\omega$ where $\tau$ is the corresponding time scale). $E(\omega)$ is thus defined as:

$$E(\omega) = \left\langle \left| \frac{f(\omega)}{\omega} \right|^2 \right\rangle$$

(1)

where $\frac{f(\omega)}{\omega}$ is the Fourier transform of $f(t)$ and the angular brackets “$\langle . \rangle$” indicate statistical averaging. $\left\langle f(t)^2 \right\rangle$ is thus the total variance (assumed to be independent of time), so that the spectral density thus satisfies

$$\left\langle f(t)^2 \right\rangle = \int \limits_0^\infty E(\omega) d\omega$$

In a scaling regime, we have power law spectra:

$$E(\omega) = \omega^{-\beta}$$

(2)

If we now consider the real space reduction in scale by factor $\lambda$, we obtain: $\omega \rightarrow \lambda^{-1} \omega$ corresponding to a “blow up” in frequencies: $\omega \rightarrow \lambda \omega$; a power law $E(\omega)$ (eq. 2) maintains its form under this transformation: $E \rightarrow \lambda^{-\beta} E$ so that $E$ is “scaling” and the “spectral exponent” $\beta$ is “scale invariant”. If empirically we find $E$ of the form eq. 2, we take this as evidence for the
scaling of the field $f$. Note that numerical spectra have well known finite size effects; the low frequency effects have been dealt with below using standard “windowing” techniques (here a Hann window was used to reduce spectral leakage).

2.2 Temporal spectral scaling in the weather regime

One of the earliest atmospheric spectral analyses was that of (Van der Hoven, 1957) whose graph is at the origin of the legendary “meso-scale gap”, the supposedly energy-poor spectral region between roughly 10-20 minutes and $\approx 4$ days (ignoring the diurnal spike). Even until fairly recently, textbooks regularly reproduced the spectrum (often redrawing it on different axes or introducing other adaptations), citing it as convincing empirical justification for the neat separation between low frequency isotropic 2D turbulence - identified with the weather - and high frequency isotropic 3-D “turbulence”. This picture was seductive since if the gap had been real, the turbulence would be no more than an annoying source of perturbation to the (2-D) weather processes.

However, it was quickly and strongly criticized (e.g. (Goldman, 1968), (Pinus, 1968), (Vinnichenko, 1969), (Vinnichenko and Dutton, 1969 ), (Robinson, 1971) and indirectly by (Hwang, 1970)). For instance, on the basis of much more extensive measurements (Vinnichenko, 1969) commented that even if the mesoscale gap really existed, it could only be for less than 5% of the time; he then went on to note that Van der Hoven’s spectrum was actually the superposition of four spectra and that the extreme set of high frequency measurements were taken during a single one hour long period during an episode of ‘near-hurricane’ conditions and these were entirely responsible for the high frequency “bump”.

More modern temporal spectra are compatible with scaling from dissipation scales to $\approx 5$-20 days. Numerous wind and temperature spectra now exist from milliseconds to hours and days showing for example that $\beta \approx 1.6, 1.8$ for $v$, $T$ respectively, some of this evidence is reviewed in (Lovejoy and Schertzer, 2010) (Lovejoy and Schertzer, 2011). Fig. 2 shows an example, the hourly temperature spectrum for frequencies down to $(4 \text{ yrs})^{-1}$. According to fig. 2, it is plausible that the scaling in the wind holds from small scales out to scales of $\approx 5 - 10$ days where we see a transition. This transition is essentially the same as the low frequency “bump” observed by Van der Hoven; its appearance only differs because he used a $\omega E(\omega)$ rather than $logE(\omega)$ plot.
Fig. 2: This shows the scaling of hourly surface temperatures from 4 stations in the northwest US, for 4 years (2005-2008) taken from the US Climate Reference Network. One can see that in spite of the strong diurnal cycle (and harmonics), the basic scaling extends to about 7 days. The reference lines (with absolute slopes 0.2, 2: these values are theoretically motivated, for low frequency weather and weather scales respectively). The spectra of hourly surface temperature data from 4 nearly colinear stations running north west - south east in the US (Lander, WY, Harrison NE, Whitman NE, Lincoln NE), from the US Climate Reference Network, 2005-2008. The yellow line is the raw spectrum, the thick line is the spectrum of the periodically detrended spectrum, averaged over logarithmically spaced bins, 10 per order of magnitude.

2.3 Temporal spectral scaling in the low frequency weather-climate regime

Except for the annual cycle, the roughly flat low frequency spectra of fig. 2 (and fig. 1 (between ≈ 10 days and 10 yrs) are qualitatively reproduced in all the standard meteorological fields and the transition scale $\tau_w$ is relatively constant. Fig. 3 shows estimates of $\tau_w$, estimated using reanalyses taken from the Twentieth Century Reanalysis project, 20CR, (Compo et al., 2011), on 2°x2° grid boxes. Also shown are estimates of $\tau_c$, the scale where the latter ends and the climate regime begins. The factor ≈ 1000 between $\tau_w$ and $\tau_c$ is the low frequency weather regime. Also shown are estimates of the planetary scale eddy turn over time discussed below.

Fig. 4 shows the surface air temperature analysis out to lower frequencies and compares this with the corresponding spectrum for sea surface temperatures (section 2.7). We see that the ocean behaviour is qualitatively similar except that the transition time scale $\tau_c$ is ≈ 1 yr and the “low frequency ocean” exponent $\beta_{lo} \approx 0.6$ which is a bit larger than the corresponding $\beta_{lw} \approx 0.2$ for the air temperatures (“lw”, “lo” for “low” frequency “weather” and “ocean” respectively). For comparison we also show the best fitting Orenstein-Uhlenbeck processes (essentially integrals of
white noises, i.e. with $\beta_w = 2, \beta_{lw} = 0$, these are the basis of Stochastic Linear Modeling approaches (e.g. (Penland, 1996)).

![Graph showing variation of $\tau_w$ (bottom curves) and $\tau_c$ (top curves) as a function of latitude. The graph compares the 138 year long 20CR reanalyses temperature field with the theoretically predicted planetary scale eddy turnover time ($\tau_{eddy}$, thick blue) and the effective external scale ($\tau_{eff}$) of the temperature cascade estimated from the ECMWF interim reanalysis for 2006 (thin blue). $\tau_w$ estimates were made by performing bi-linear regressions on spectra from 180 day long segments averaged over 280 segments per grid point. The $\tau_c$ were estimated by bi-linear fits on the Haar structure functions applied to the same data but monthly averaged.]

Fig. 3: The variation of $\tau_w$ (bottom curves) and $\tau_c$ (top curves) as a function of latitude as estimated from the 138 year long 20CR reanalyses, 700 mb temperature field compared with the theoretically predicted planetary scale eddy turnover time ($\tau_{eddy}$, thick blue) and the effective external scale ($\tau_{eff}$) of the temperature cascade estimated from the ECMWF interim reanalysis for 2006 (thin blue). The $\tau_w$ estimates were made by performing bi-linear regressions on spectra from 180 day long segments averaged over 280 segments per grid point. The $\tau_c$ were estimated by bi-linear fits on the Haar structure functions applied to the same data but monthly averaged.

To underline ubiquity of the low frequency weather regime, its low $\beta$ character and to distinguish it from the higher frequency weather regime, this regime was called the “spectral plateau” (Lovejoy and Schertzer, 1986), although it is somewhat of a misnomer since it is clear that the regime has a small but nonzero logarithmic slope whose negative value we indicate by $\beta_{lw}$. The transition scale $\tau_w$ was also identified as a weather scale by (Koscielny-Bunde et al., 1998).
Fig. 4: This figure superposes the ocean and atmospheric plateaus showing their great similarity.

**Left:** A comparison of the monthly SST spectrum (bottom, blue) and monthly atmospheric temperatures over land (top, purple) for monthly temperature series from 1911-2010 on a 5°x5° grid; the NOAA NCDC data). Only those near complete series (missing less than 20 months out of 1200) were considered; 465 for the SST, 319 for the land series; the missing data were filled using interpolation. The reference slopes correspond to $\beta = 0.2$ (top), 0.6 bottom left and 1.8, bottom right. A transition at 1 year corresponds to a mean ocean $\varepsilon_o \approx 1 \times 10^{-8} m^2 s^{-3}$. The dashed orange lines are Orenstein-Uhlenbeck processes (of the form $E(\omega) = \sigma^2 / (\omega^2 + a^2)$, $\sigma$, a are constants) used as the basis for Stochastic Linear Forcing models.

**Right:** The average of 5 spectra from 6 year long sections of a thirty year series of daily temperatures at a station in France (black, taken from (Lovejoy and Schertzer, 1986)). The red reference line has a slope 1.8. The relative up-down placement of this daily spectrum with the monthly spectra (corresponding to a constant factor) was determined by aligning the atmospheric spectral plateaus (i.e. the black and purple spectra). The raw spectra are shown (no averaging over logarithmically spaced bins).

### 2.4 The weather regime, space-time scaling and some turbulence theory

In order to understand the weather and low frequency weather scaling, we briefly recall some turbulence theory using the example of the horizontal wind $v$. In stratified scaling turbulence (the 23/9D model (Schertzer and Lovejoy, 1985a)), the energy flux $\varepsilon$ dominates the horizontal and the buoyancy variance flux $\phi$ dominates the vertical so that horizontal wind fluctuations $\Delta v$ (e.g. differences, see section 3) follow:
\[ \Delta v(\Delta x) = \varepsilon^{1/3} \Delta x^{H_v}; \quad H_v = 1/3 \quad 3a \]

\[ \Delta v(\Delta y) = \varepsilon^{1/3} \Delta y^{H_v}; \quad H_v = 1/3 \quad 3b \]

\[ \Delta v(\Delta z) = \phi^{1/5} \Delta z^{H_v}; \quad H_v = 3/5 \quad 3c \]

\[ \Delta v(\Delta t) = \varepsilon^{1/2} \Delta t^{H_v}; \quad H_v = 1/2 \quad 3d \]

where \( \Delta x, \Delta y, \Delta z, \Delta t \) are the increments in horizontal coordinates, vertical coordinate and time respectively and the exponents \( H_h, H_v, H_z \) are “fluctuation” or “non conservation” exponents in the horizontal, vertical and in time respectively. Since the mean fluxes are independent of scale (i.e. \( <\varepsilon>, <\phi> \) are constant, “\( <,> \)” indicates ensemble averaging) these exponents express how the fluctuations \( \Delta v \) increase \( (H>0) \) or decrease \( (H<0) \) with the scale (i.e. the increments). Equations \( (3 \ a - b) \) describe the real space horizontal (Kolmogorov, 1941) scaling and \( 3 \ c \) the vertical Bolgiano-Obukhov (BO, (Bolgiano, 1959), (Obukhov, 1959)) scaling for the velocity. The anisotropic Corrsin-Obukov law for passive scalar advection is obtained by the replacements \( \nu \rightarrow \rho; \quad \varepsilon \rightarrow \chi^{3/2} \varepsilon^{-1/2} \) where \( \rho \) is the passive scalar density, \( \chi \) is the passive scalar variance flux (Corrsin, 1951), (Obukhov, 1949).

Before proceeding, a few technical comments. In eq. 3, the equality signs should be understood in the sense that each side of the equation has the same scaling properties: the Fractionally Integrated Flux model is essentially a more precise interpretation of the equations in terms of fractional integrals of order \( H \). Ignoring intermittency (associated with multifractal fluxes which we discuss only briefly below), the spectral exponents are related to \( H \) as \( \beta =1+2H \) so that \( \beta_h =5/3, \beta_v =11/5, \beta_z=2 \). Finally, although the notation “\( H' \)” is used in honour of E. Hurst, in multifractal processes it is generally not identical to the Hurst (i.e. rescaled range, “R/S”) exponent, the relationship between the two is nontrivial.

Although these equations originated in classical turbulence theory, the latter were all spatially statistically isotropic so that the simultaneous combination of the horizontal laws \( 3a, b \) with the vertical law \( 3c \) is nonclassical. Since the isotropy assumption is very demanding, the pioneers originally believed that the classical laws would hold over only scales of scales of hundreds of meters. The anisotropic extension implied by eqs. 3 a-d is itself based on a generalization of the notion of scale invariance - thus has the effect of radically changing the potential range of validity of the laws. For example, even the finite thickness of the troposphere - which in isotropic turbulence would imply a scale break at around 10 \( km \) – no longer implies a break in the scaling. Beginning with (Schertzer and Lovejoy, 1985b), it has been argued that atmospheric variables including the wind do indeed have wide range (anisotropic) scaling statistics (see the review (Lovejoy and Schertzer, 2010)).

In addition, the classical turbulence theories were for spatially uniform (“homogeneous”) turbulence in which the fluxes were quasi constant (e.g. with Gaussian statistics). In order for these laws to apply up to planetary scales, starting in the 1980’s, (Parisi and Frisch, 1985), (Schertzer and Lovejoy, 1985b) they were generalized to strongly variable (intermittent)
multiplicative cascade processes yielding multifractal fluxes so that although the mean flux statistics \( \langle \epsilon_{\Delta x} \rangle \) remain independent of scale \( \Delta x \), the statistical moments:

\[
\langle \epsilon_{\Delta x}^q \rangle \approx \Delta x^{K(q)}
\]

(4) where \( \epsilon_{\Delta x} \) is the flux averaged over scales \( \Delta x \) and \( K(q) \) is the (convex) moment scaling exponent.

Although \( K(1) = 0 \), for \( q \neq 1 \), \( K(1) \neq 0 \) so that \( \langle \epsilon_{\Delta x}^q \rangle \) is strongly scale dependent, the fluxes are thus the densities of singular multifractal measures.

Along with the spatial laws (eqs. 3 a-c), we have included eq. 3 d which is the result for the pure time evolution in the absence of an overall advection velocity; this is the classical Lagrangian version of the Kolmogorov law (Inoue, 1951; Landau and Lifschitz, 1959), it is essentially the result of dimensional analysis using \( \epsilon \) and \( \Delta t \) rather than \( \epsilon \) and \( \Delta x \). Although Lagrangian statistics are notoriously difficult to obtain empirically (see however (Seuront et al., 1996)), they are roughly known from experience and are used as the basis for the space-time or “Stommel” diagrams that adorn introductory meteorology textbooks (see (Schertzer et al., 1997), (Lovejoy et al., 2000) for scaling adaptations).

Due to the fact that the wind is responsible for advection, the spatial scaling of the horizontal wind leads to the temporal scaling of all the fields. Unfortunately, space-time scaling is somewhat more complicated than pure spatial scaling. At meteorological time scales this is because we must take into account the mean advection of structures and the Galilean invariance of the dynamics. The effect of the Galilean invariance / advection of structures is that the temporal exponents in the Eulerian (fixed earth) frame become are the same as in the horizontal direction (i.e. in \((x,y,t)\) space we have “trivial anisotropy”, i.e. with “effective” temporal exponent \( H_{\text{eff}} = H_b \)). At the longer time scales of low frequency weather, the scaling is broken because the finite size of the earth implies a characteristic lifetime (“eddy-turn-over time”) of planetary scale structures. Using eq. 3 a,b with \( \Delta x = L_e \) we obtain: 

\[
\tau_{\text{eddy}} = L_e/\Delta v(L_e) = \epsilon_w^{-1/3} L_e^{2/3}
\]

where \( L_e = 20,000 \text{ km} \) is the size of the planet and \( \epsilon_w \) is the globally averaged energy flux density.

Considering time scales longer than \( \tau_{\text{eddy}} \), the effect of the finite planetary size implies that the spatial degrees of freedom become ineffective - there is a “dimensional transition” – so that instead of interactions in \((x,y,z,t)\) space for long times, the interactions are effectively only in \( t \) space and this implies a drastic change in the statistics, summarized in the next section.

### 2.5 Low frequency weather and a dimensional transition

To obtain theoretical predictions for the statistics of atmospheric variability at time scales \( \tau > \tau_w \) (i.e. in the low frequency weather regime), we can take the FIF model that produces multifractal fields respecting eqs. 3 and extend it to processes with outer timescales \( \tau >> \tau_w \). The theoretical details are given in (Lovejoy and Schertzer, 2010) and (Lovejoy and Schertzer, 2011) but the upshot of this is that we expect the energy flux density \( \epsilon \) to factor into a statistically independent space-time weather process \( \epsilon_w(r,t) \) and a low frequency weather process \( \epsilon_{l\omega}(t) \) which is only dependent on time:

\[
\epsilon(r,t) = \epsilon_w(r,t) \epsilon_{l\omega}(t)
\]

(5)
In this way the low frequency energy flux $\varepsilon_{lw}(t)$ - which physically is the result of nonlinear radiation/cloud interactions – multiplicatively modulates the high frequency space-time weather processes.

The statistical behaviour of $\varepsilon_{lw}(t)$ is quite complex to analyze and has some surprising properties. Some important characteristics are: a) at large temporal lags $\Delta t$ the autocorrelations $<\varepsilon_{lw}(t)\varepsilon_{lw}(t-\Delta t)>$ ultimately decay as $\Delta t^{-1}$, although very large ranges of scale may be necessary to observe it, b) since the spectrum is the Fourier transform of the autocorrelation and the transform of a pure $\Delta t^{-1}$ function has a low (and high) frequency divergence, the actual spectrum of the low frequency weather regime depends on its overall range of scales $\Lambda_c=\tau_c/\tau_w$; from fig. 3, we find that to within a factor of $\approx 2$, that the mean $\Lambda_c$ over the latitudes $\approx 1100$, c) over surprisingly wide ranges (factors of 100 – 1000 in frequency for values of $\Lambda_c$ in the range $2^{10}$ - $2^{16}$), one finds “pseudo-scaling” with nearly constant spectral exponents $\beta_{lw}$ which are typically in the range $0.2 – 0.4$, d) the statistics are independent of $H$ and only weakly dependent on $K(q)$.

In summary, we therefore find for the overall turbulence (FIF) model:

$$E(k) = k^{-\beta_w}; \quad k > \Lambda_c$$
$$E(\omega) \approx \omega^{-\beta_w}; \quad \omega > \tau_w^{-1}$$
$$E(\omega) \approx \omega^{-\beta_w}; \quad \tau_c^{-1} < \omega < \tau_w^{-1}$$

(6)

where $k$ is the modulus of the horizontal wave vector, $\tau_c$ is the long external scale where the low frequency weather regime ends (see discussion below) and $\beta_l$, $\beta_{lw}$ are:

$$\beta_w = 1 + 2H_w - K(2)$$
$$0.2 < \beta_{lw} < 0.6$$

(7)

In the low frequency weather regime, the intermittency (characterized by $K(q)$) decreases as we average the process over scales $>\tau_w$ so that the low frequency weather regime has an effective fluctuation exponent $H_{lw}$:

$$H_{lw} \approx -(1-\beta_{lw})/2$$

(8)

The high frequency weather spectral exponents $\beta_w$, $H_w$ are the usual ones, but the low frequency weather exponents $\beta_{lw}$, $H_{lw}$ are new. Since $\beta_{lw}<1$, we have $H_{lw}<0$ and using $0.2 < \beta_{lw} < 0.6$ corresponding to -0.4 $< H_{lw} < -0.2$, this result already explains the preponderance of spectral plateau $\beta$’s around that value already noted. However, as we saw in fig. 4, the low frequency ocean regime has a somewhat high value $\beta_{oc} \approx 0.6$ ($H_{oc} \approx -0.2$); (Lovejoy and Schertzer, 2011) use a simple coupled ocean - atmosphere model to show how this could arise as a consequence of double (atmosphere and ocean) dimensional transitions.
2.6 The transition time scale from weather to low frequency weather using “first principles”

Figures 1 – 4 show evidence that temporal scaling holds from small scales to a transition scale \( \tau_w \) of around 5 -20 days. Let us now consider the physical origin of this scale. In the famous (Van der Hoven, 1957) \( \omega E(\omega) \) versus \( \log \omega \) plot, its origin was argued to be due to “migratory pressure systems of synoptic weather- map scale”. The corresponding features at around 4 – 20 days notably for temperature and pressure spectra were termed “synoptic maxima” by (Kolesnikov and Monin, 1965), and (Panofsky, 1969) in reference to the similar idea that it was associated with synoptic scale weather dynamics, see (Monin and Yaglom, 1975) for some other early references.

More recently, (Vallis, 2010) suggested that \( \tau_w \) is the basic lifetime of baroclinic instabilities which he estimated using the inverse Eady growth rate: \( \tau_{Eady} = L_d / U \) where the deformation rate is: \( L_d = \sqrt{Ri} / f_0 \), \( f_0 \) is the Coriolis parameter and \( Ri \) is the Richardson number across the troposphere (= \( N^2/s^2 \) where \( N \) is the mean Brunt-Vaisalla frequency and \( s \) the mean shear). The Eady growth rate is obtained by linearizing the equations about a hypothetical state with both uniform shear and stratification across the troposphere. By taking \( Ri \approx 10^4 \), \( f_0 \approx 10^4 \) \( rad/s \), and \( U \approx 10 \ m/s \), one obtain Vallis’s estimate \( L_d \approx 1000 \ km \). Using the maximum Eady growth rate theoretically introduces a numerical factor 3.3 so that the actual predicted inverse growth rate is: \( 3.3 \tau_{Eady} \approx 4 \ days \). Vallis similarly argues that this also applies to the oceans but with \( U \approx 10 \ cm/s \) and \( L_d \approx 100 \ km \) yielding \( 3.3 \tau_{Eady} \approx 40 \ days \). The obvious theoretical problem with using \( \tau_{Eady} \) to estimate \( \tau_w \), is that the former is expected to be valid in homogeneous, quasi-linear systems whereas the atmosphere is highly heterogeneous with vertical and horizontal structures (including strongly nonlinear cascade structures) extending throughout the troposphere to scales substantially larger than \( L_d \). Another difficulty is that although the observed transition scale \( \tau_w \) is well behaved at the equator (fig. 3), \( f_0 \) vanishes implying that \( L_d \) and \( \tau_{Eady} \) diverge: using \( \tau_{Eady} \) as an estimate of \( \tau_w \) is at best a midlatitude approximation. Finally, the choice of values used to estimate \( \tau_{Eady} \) are not obvious. For example, the estimate \( Ri \approx 10^4 \) across the troposphere seems quite high, and there is no evidence for any special behaviour at length scales near \( L_d \approx 1000 \ km \).

If there is (at least statistically) a well defined relation between spatial scales and lifetimes (the “eddy-turn-over time”), then the lifetime of planetary scale structures \( \tau_{eddy} \) is of fundamental importance. The shorter period (\( \tau < \tau_{eddy} \)) statistics are dominated by structures smaller than planetary size whereas for \( \tau > \tau_{eddy} \), they are dominated by the statistics of many lifetimes of planetary scale structures. It is therefore natural to take \( \tau_{eddy} \approx \tau_w \).

In order to estimate \( \tau_w \) we therefore need an estimate of the globally averaged flux energy density \( \epsilon_w \). We can estimate \( \epsilon_w \) by using the fact that the mean solar flux absorbed by the earth is \( \approx 200 \ W/m^2 \) (e.g. (Monin, 1972)). If we distribute this over the troposphere (thickness \( \approx 10^4 m \)), with mean air density \( \approx 0.75 \ Kg/m^3 \), and we assume a 2% conversion of energy into kinetic energy ((Palmén, 1959), (Monin, 1972)), then we obtain a value \( \epsilon_w \approx 5 \times 10^4 m^2/s^3 \) which is indeed typical of the values measured in small scale turbulence (Brunt, 1939), (Monin, 1972).
Using the ECMWF interim reanalysis to obtain a modern estimate of $\varepsilon_o$ (Lovejoy and Schertzer, 2010) showed that although $\varepsilon$ is larger in mid latitudes than at the equator and that at 300 mb it reaches a maximum, the global tropospheric average is $\approx 10^{\frac{5}{3}} \text{m}^2/\text{s}^3$. They also showed that the latitudinally varying $\varepsilon$, explained to better than $\pm 20\%$ the latitudinal variation of the hemispheric antipodes velocity differences (using $\Delta v = \varepsilon^{1/3} L_o^{1/3}$) and concluded that the solar energy flux does a good job of explaining the horizontal wind fluctuations up to planetary scales.

In addition, we can now point to fig. 3 which shows that the latitudinally varying ECMWF $\varepsilon_o(\theta)$ estimates do indeed lead to $\tau_{\text{eddy}}(\theta)$ very close to the 20CR $\tau_o(\theta)$ estimates for the 700 mb temperature field.

### 2.7 Ocean “weather” and “low frequency ocean weather”

It is well known that for months and longer time scales that ocean variability is important for atmospheric dynamics; before explicitly attempting to extend this model of weather variability beyond $\tau_o$, we must therefore consider the role of the ocean. The ocean and the atmosphere have many similarities; from the preceding discussion, we may expect analogous regimes of “ocean weather” to be followed by an ocean spectral plateau both of which will influence the atmosphere. To make this more plausible, recall that both the atmosphere and ocean are large Reynolds’ number turbulent systems and both are highly stratified - albeit due to somewhat different mechanisms. In particular there is no question that at least over some range, horizontal ocean current spectra are dominated by the ocean energy flux $\varepsilon_o$. It roughly follows that $E(\omega) \approx \omega^{-5/3}$ and presumably in the horizontal: $E(k) \approx k^{-5/3}$; (i.e. $\beta_o = 5/3$, $H_o = 1/3$, see e.g. (Grant et al., 1962), (Nakajima and Hayakawa, 1982)). Although surprisingly few current spectra have been published, the recent use of satellite altimeter data to estimate sea surface height (a pressure proxy) has provided relevant empirical evidence that $k^{-5/3}$ continues out to scales of at least hundreds of kilometers, refuelling the debate about the spectral exponent and the scaling of the current, see (Le Traon et al., 2008).

Although empirically the current spectra (or their proxies) at scales larger than several hundred kilometres are not well known, other spectra - especially those of sea surface temperatures (SST) - are known to be scaling over wide ranges and due to their strong nonlinear coupling with the current they are relevant. Using mostly remotely sensed infrared radiances, and starting in the early 1970’s, there is much evidence for SST scaling up to thousands of kilometres with $\beta \approx 1.8$ -2 i.e. nearly the same as for the atmospheric temperature (see e.g. (McLeish, 1970), (Saunders, 1972), (Deschamps et al., 1981), (Deschamps et al., 1984), (Burget and Hsieh, 1989), (Seuront et al., 1996), (Lovejoy et al., 2000), and a review in (Lovejoy and Schertzer, 2011)).

If, as in the atmosphere, the energy flux dominates the horizontal ocean dynamics then we can use the same methodology as in the previous subsection - basic turbulence theory (the Kolmogorov law) combined with the mean ocean energy flux $\varepsilon_o$ - to predict ocean eddy turn over time and hence the outer scale $\tau_o$ of the ocean regime. Thus, for ocean gyres and eddies of size $l$, we expect there to be a characteristic eddy turnover time (lifetime) $\tau = \varepsilon^{-1/3} l^{2/3}$ with a critical “ocean-weather” – “ocean-climate” transition time scale obtained when $l = L_o$: $\tau_o = \varepsilon_o^{-1/3} L_o^{2/3}$.

Again, we expect a fundamental difference in the statistics for fluctuations of duration $\tau< \tau_o$ - the ocean equivalent of “weather” with a turbulent spectrum with roughly $\beta_o \approx 5/3$ (at least for the
current) - and for durations \( \tau > \tau_c \), the ocean “climate” with a shallow ocean spectral regime with \( \beta \approx 1 \). Since the spatial \( \beta \) for temperature in the atmosphere and ocean are very close, if the \( \beta \)'s for the current and wind are also close, then so will the \( \beta \) for the temporal temperature spectra.

In order to test this idea, we need the globally averaged ocean current energy flux, \( \varepsilon_o \). As expected, \( \varepsilon_o \) is highly intermittent (see (Robert, 1976), (Clayson and Kantha, 1999), (Moum et al., 1995), (Lien and D’Asaro, 2006), (Matsuno et al., 2006)) and as far as we know, the only attempt to estimate its global average is (Lovejoy and Schertzer, 2011) who used ocean drifter maps of eddy kinetic energy. They found that \( \varepsilon_o \approx 10^8 \text{ m}^2/\text{s}^3 \) is a reasonable global estimate for the surface layer (it decreases quite rapidly with depth). Using the formula \( \tau_o = \varepsilon_o^{-1/3}L^2/3_c \) and \( \varepsilon_o \) in the range \( 1 \times 10^8 - 8 \times 10^8 \text{ m}^2/\text{s}^3 \) we find \( \tau_o \approx 1 - 2 \text{ yrs} \); c.f. the values for the atmosphere (\( \varepsilon_o \approx 10^3 \text{ m}^2/\text{s}^3 \)), \( \tau_o \approx 10 \text{ days} \).

This provides us with a prediction for the SST spectrum: \( E(\omega) \approx \omega^{1.8} \) for \( \omega \approx \geq (1 \text{ year})^{-1} \) followed by a transition to a much flatter plateau (here \( \approx \omega^{0.6} \)) for the lower frequencies, see fig. 6 which compares the ocean and air over land spectra. While the latter spectrum is – as expected - essentially a pure spectral plateau (with \( \beta_{lw} \approx 0.2 \), the value cited earlier), we see that the SST spectrum is essentially the same (\( \beta_o \approx \beta_u \approx 1.8 \)) except that \( \beta_o \approx 0.6 \) and \( \tau_o \approx 1 \text{ year} \). This basic “cross-over” to an exponent \( \beta_o \approx 0.6 \) was already noted by (Monetti et al., 2003) who estimated it at 300 days. Note also the rough convergence of the spectra at about 100 year scale which implies that the land and ocean variability become equal and also the hint that there is a low frequency rise in the land spectrum for periods \( \approx 30 \text{ yrs} \).

### 2.8 Other evidence for the spectral plateau

Various published scaling composites such as figs. 1a, b give estimates for the low frequency weather exponent \( \beta_{lw} \), the climate exponent \( \beta_c \), and the transition scale \( \tau_c \); they agree on the basic picture while proposing somewhat different parameter values and transition scales \( \tau_c \). For example (Huybers and Curry, 2006a) studied many paleoclimate series as well as the 60 year long NCEP reanalyses concluded that for periods of months up to about 50 years, the spectra are scaling with midlatitude \( \beta_{lw} \)'s larger than the tropical \( \beta_{lw} \) (their values are \( 0.37 \pm 0.05, 0.56 \pm 0.08 \)). Many analyses in the spectral plateau regime have been carried out using in situ data with the Detrended Fluctuation Analysis (DFA) method (Fraedrich and Blender, 2003; Koscielny-Bunde et al., 1998), (Bunde et al., 2004), sea surface temperatures (Monetti et al., 2003) and \( \approx 1000 \) year long Northern hemisphere reconstructions (Rybski et al., 2006) (see also Lennartz and Bunde, 2009) and (Lanfredi et al., 2009); see (Eichner et al., 2003), for a review of many scaling analyses and their implications for long term persistence / memory issue. From the station analyses the basic conclusions of (Fraedrich and Blender, 2003) were that over land, \( \beta \approx 0-0.1 \) whereas over the ocean \( \beta \approx 0.3 \); whereas (Eichner et al., 2003) found \( \beta \approx 0.3 \) over land and using NCEP reanalyses, (Huybers and Curry, 2006a) found slightly higher values and noted an additional latitudinal effect (\( \beta \) is higher at the equator). At longer scales, (Blender et al., 2006) analysed the anomalous Holocene Greenland paleo temperatures finding \( \beta \approx 0.5 \), see section 4.2. Other pertinent analyses are of Global Climate Model outputs and historical reconstructions of the Northern hemisphere temperatures which are discussed in detail in section 4.3. Our basic empirical conclusions, in accord with a growing literature – particularly
with respect to the temperature statistics are that $\beta$ is mostly in range 0.2 – 0.4 over land and $\approx$0.6 over the ocean.

3. Climate change

3.1 What is climate change?

We briefly surveyed the weather scaling, focusing on the transition to the low frequency weather regime for time scales longer than the lifetimes of planetary scale eddies at $\tau_w \approx 5$ - 20 days. This picture was complicated somewhat by the qualitatively similar (and nonlinearly coupled) transition from the analogous ocean “weather” to “low frequency ocean weather” at $\tau_o \approx 1$ yr. Using purely spectral analyses, we found that these low frequency regimes continued until scales of the order of $\tau_c \approx 10$ - 100 years, after which the spectra started to steeply rise, marking the beginning of the true climate regime. While the high frequency regime clearly corresponds to “weather”, we termed the intermediate regime “low frequency weather” since its statistics are not only well reproduced with (unforced) “control” runs of GCM’s (fig. 1b) but also by (stochastic, turbulent) cascade models of the weather when they are extended to low frequencies. The term “climate regime” was thus reserved for the long times $\tau > \tau_c$ where the low frequency weather regime gives way to a qualitatively different and much more variable regime. The new climate regime is thus driven either by new (internal) low frequency nonlinear interactions, or by appropriate low frequency solar, volcanic, anthropogenic - or eventually orbital - forcing at scales $\tau > \tau_c$.

This three scale-range scaling picture of atmospheric variability leads to a clarification of the rough idea that the climate is nothing more than long-term averages of the weather. It allows us to precisely define a climate state as the average of the weather over the entire low frequency weather regime up to $\tau_c$ (i.e. up to decadal or centennial scales). This paves the way for a straightforward definition of climate change as the long term changes in this climate state i.e. of the statistics of these climate states at scales $\tau > \tau_c$.

3.2 What is $\tau_c$?

In figure 1, we gave some evidence that $\tau_c$ was in the range (10 yr)$^{-1}$ to (100 yrs)$^{-1}$, i.e. it was near the extreme low frequency limit of instrumental data. We now attempt to answer this with more certainty. Up until now, we primarily used spectral analysis since it is a classical, straightforward technique whose limitations are well known and it was adequate for the purpose of determining the basic scaling regimes in time and in space. We now focus on the low frequencies corresponding to several years to $\approx 100$ kyrs so that it is convenient to study fluctuations in real rather than Fourier space. There are several reasons for this. The first is that we are focusing on the lowest instrumental frequencies, and so spectral analysis provides only a few useful data points - for example on data 150 years long, the time scales longer than 50 years are characterized only by three discrete frequencies $\omega =$1, 2, 3: Fourier methods are
“coarse” at low frequencies. The second is that in order to extend the analysis to lower frequencies it is imperative to use proxies, and these need calibration: the mean absolute amplitudes of fluctuations at a given scale enable us to perform a statistical calibration. The third is that the absolute amplitudes are also important for gauging the physical interpretation and hence significance of the fluctuations.

### 3.3 Fluctuations and structure functions

The simplest fluctuation is also the oldest, the difference: \((\Delta v(\Delta t))_{\text{diff}} = \Delta v(t+\Delta t) - \Delta v(t)\). The corresponding statistical moments \(<\Delta v^q>\) are the classical “generalized” structure functions. According to eq. 3, the fluctuations follow:

\[
\Delta v = \varphi_{\Delta t} \Delta t^H
\]

where \(\varphi_{\Delta t}\) is a resolution \(\Delta t\) turbulent flux. From this we see that the statistical moments follow:

\[
\left< \Delta v(\Delta t)^q \right> = \left< \varphi_{\Delta t}^q \right> \Delta t^{qH} = \Delta t^{q(\xi(q))}; \quad \xi(q) = qH - K(q)
\]

\(\xi(q)\) is the structure function with exponent and \(K(q)\) is the (multifractal, cascade) intermittency exponent, eq. 4. The turbulent flux has the property that it is independent of scale \(\Delta t\), i.e. the first order moment \(<\varphi_{\Delta t}>\) is constant, hence \(K(1) = 0\) and \(\xi(1) = H\). The physical significance of \(H\) is thus that it determines the rate at which fluctuations grow \((H>0)\) or decrease \((H<0)\) with scale \(\Delta t\).

Since the spectrum is a second order moment, there is the following useful and simple relation between real space and Fourier space exponents:

\[
\beta = 1 + \xi(2) = 1 + 2H - K(2)
\]

The problem is that the mean difference cannot decrease with increasing \(\Delta t\), hence differences are clearly inappropriate when studying scaling processes with \(H<0\): the differences simply converge to a spurious constant depending on the highest frequencies available in the sample. Similarly, when \(H>1\), fluctuations defined as differences saturate at a large \(\Delta t\) independent value; they depend on the lowest frequencies present in the sample. In both cases, the exponent \(\xi(q)\) is no longer correctly estimated. The problem is that we need a definition of fluctuations such that \(\Delta v(\Delta t)\) is dominated by frequencies \(\approx \Delta t^{-1}\).

The need to more flexibly define fluctuations motivated the development of wavelets (e.g. (Bacry et al., 1989), (Mallat and Hwang, 1992; Torrence and Compo, 1998)), and the related Detrended Fluctuation Analysis technique (DFA, (Peng et al., 1994) and (Kantelhardt et al., 2001), (Kantelhardt et al., 2002) for polynomial and multifractal extensions respectively. In this context, the classical difference fluctuation is only a special case, the “poor man’s wavelet”. In the weather regime, most geophysical \(H\) parameters are indeed in the range 0 to 1 (see e.g. the review (Lovejoy and Schertzer, 2010)) so that fluctuations tend to increase with scale and this classical difference structure function is generally adequate. However, a prime characteristic of the low frequency weather regime is precisely that \(H<0\) (section 2.5) so that fluctuations decrease rather than increase with scale, hence for studying this regime, difference fluctuations are inappropriate. To change the range of \(H\) over which fluctuations are usefully defined, one changes the shape of the defining wavelet, changing both its real and Fourier space
localizations. In the usual wavelet framework, this is done by modifying the wavelet directly
e.g. by choosing the Mexican hat or higher order derivatives of the Gaussian etc., or by choosing
them to satisfy some special criterion. Following this, the fluctuations are calculated as
convolutions with Fast Fourier (or equivalent) numerical techniques.

A problem with this usual implementation of wavelets is that not only are the
convolutions numerically cumbersome, but the physical interpretation of the fluctuations is lost.
In contrast, when \(0<H<1\), the difference structure function gives direct information on the typical
difference \((q =1)\) and typical variations around this difference \((q =2)\) and even typical skewness
\((q =3)\) or typical Kurtosis \((q =4)\) or - if the probability tail is algebraic – of the divergence of high
order moments of differences. Similarly, when \(-1<H<0\) one can define the “tendency structure
function” (below) which directly quantifies the fluctuation’s deviation from zero and whose
exponent characterizes the rate at which the deviations decrease when we average to larger and
larger scales. These poor man’s and tendency fluctuations are also very easy to directly estimate
from series with uniformly spaced data and - with straightforward modifications - to irregularly
spaced data.

The study of real space fluctuation statistics in the low frequency weather regime
therefore requires a definition of fluctuations valid at least over the range -1\(<H<1\). Before
discussing our choice - the Haar wavelet - let us recall the definitions of the difference and
tendency fluctuations; the corresponding structure functions are simply the \(q^{th}\) moments. The
difference/ poor man’s fluctuation is thus:

\[
(\Delta v(\Delta t))_{\text{diff}} = |\delta_{\Delta t} v|; \quad \delta_{\Delta t} v = v(t + \Delta t) - v(t)
\]

(12)

where \(\delta\) is the difference operator. Similarly, the “tendency fluctuation” (Lovejoy and
Schertzer, 2011) can be defined using the series with overall mean removed:

\[
v'(t) = v(t) - \overline{v(t)}
\]

with the help of the summation operator \(s\) by:

\[
(\Delta v(\Delta t))_{\text{tend}} = \left| \frac{1}{\Delta t} \delta_{\Delta t} s v' \right|; \quad s v' = \sum_{t' < t} v'(t')
\]

(13)

\((\Delta v(\Delta t))_{\text{tend}}\) has a straightforward interpretation in terms of the mean tendency of the data but is
useful only for \(-1<H<0\). It is also easy to implement: simply remove the overall mean and then
take the mean over intervals \(\Delta t\): this is equivalent to taking the mean of the differences of the
running sum.

We can now define the Haar fluctuation which is a special case of the Daubechies family
of orthogonal wavelets (see e.g. (Holschneider, 1995), for a recent application, (Ashok et al.,
2010), and for a comparison with the related Detrended Fluctuation Analysis technique, see
(Koscielny-Bunde et al., 1998; Koscielny-Bunde et al., 2006)). This can be done by taking the
second differences of the mean:

\[
(\Delta v(\Delta t))_{\text{Haar}} = \frac{2}{\Delta t} \delta_{\Delta t/2} s^2 = \left| \frac{1}{\Delta t} \left( (s(t) + s(t + \Delta t)) - 2s(t + \Delta t / 2) \right) \right|
\]

\[
= \frac{2}{\Delta t} \left[ \sum_{t' < t + \Delta t / 2} v(t') - \sum_{t' < t + \Delta t / 2} v(t') \right]
\]

(14)
From this, we see that the Haar fluctuation at resolution $\Delta t$ is simply the first difference of the series degraded to resolution $\Delta t/2$. Although this is still a valid wavelet (although with the extra normalisation factor $\Delta t^1$), it is almost trivial to calculate and (thanks to the summing) the technique is useful for series with $-1<H<1$.

For pure scaling functions, the difference ($1>H>0$) or tendency ($-1<H<0$) structure functions are adequate, and have obvious interpretations. The real advantage of the Haar structure function is apparent for functions with two or more scaling regimes - one with $H>0$, one with $H<0$. We shall see that ignoring intermittency, this criterion is the same as $\beta<1$ or $\beta>1$ hence (see e.g. fig. 1a) Haar fluctuations will be useful for the data analyzed which straddle - either at high or low frequencies - the boundaries of the low frequency weather regime.

Is it possible to “calibrate” the Haar structure function so that the amplitude of typical fluctuations can still be easily interpreted? To answer this, consider the definition of a “hybrid” fluctuation as the maximum of the difference and tendency fluctuations:

$$ (\Delta T)_{\text{hybrid}} = \max \left( (\Delta T)_{\text{diff}}, (\Delta T)_{\text{trend}} \right) $$

The “hybrid structure function” is thus the maximum of the corresponding difference and tendency structure functions and therefore has a straightforward interpretation. The hybrid fluctuation is useful if a calibration constant $C$ can be found such that:

$$ \left\langle (\Delta T)^q \right\rangle_{\text{hybrid}} \approx C^q \left\langle (\Delta T)^q \right\rangle_{\text{Haar}} $$

In a pure scaling process with $-1<H<1$, this is clearly possible since the difference or tendency fluctuations yield the same scaling exponent. However, in a case with two or more scaling regimes, this equality cannot be exact, but as we see this in the next section, it can still be quite reasonable approximation.

### 3.5 Application of Haar fluctuations to global temperature series

Now that we have defined the Haar fluctuations and corresponding structure function, we can use it to analyse a fundamental climatological series: the monthly resolution global mean surface temperature. At this resolution, the high frequency weather variability is largely filtered out and the statistics are dominated first by the low frequency weather regime ($H<0$), and then at low enough frequencies by the climate regime ($H>0$).

Several such series have been constructed. The three we chose are the NOAA NCDC (National Climatic Data Center) merged land air and sea surface temperature dataset (abbreviated NOAA NCDC below, from 1880 on a 5°x5° grid, see (Smith et al., 2008) for details), the NASA GISS (Goddard Institute for Space Studies) data set (from 1880 on a 2°x2° (Hansen et al., 2010) and the HadCRUT3 data set (from 1850 to 2010 on a 5° x5° grid).

HadCRUT3 is a merged product created out of the HadSST2 (Rayner et al., 2006) Sea Surface Temperature (SST) data set and its companion data set CRUTEM3 of atmospheric temperatures over land. The NOAA NCDC and NASA GISS are both heavily based on the Global Historical Climatology Network (Peterson and Vose, 1997), and have many similarities including the use of sophisticated statistical methods to smooth and reduce noise. In contrast, the HadCRUTM3 data is less processed. Unsurprisingly, these series are quite similar although analysis of the scale by scale differences between the spectra is interesting, see (Lovejoy and Schertzer, 2011).
Each grid point in each data set suffered from missing data points so that here we consider the globally averaged series obtained by averaging over all the available data for the common 129 yrs period 1880 – 2008. Before analysis, each series was periodically detrended to remove the annual cycle – if this is not done, then the scaling near $\Delta t \approx 1 \text{ yr}$ will be artificially degraded. The detrending was done by setting the amplitudes of the Fourier components corresponding to annual periods to the “background” spectral values.

Fig. 5 shows the comparison of the difference, tendency, hybrid and Haar root mean square (RMS) structure functions $< \Delta T(\Delta t)^2 >^{1/2}$, the latter increased by a factor $C = 10^{0.35} \approx 2.2$. Before commenting on the physical implications, let us first make some technical remarks. It can be seen that the “calibrated” Haar and hybrid structure functions are very close; the deviations are $\pm 14\%$ over the entire range of nearly a factor $10^3$ in $\Delta t$. This implies that the indicated amplitude scale of the calibrated Haar structure function in degrees $K$ is quite accurate, and that to a good approximation, the Haar structure function can preserve the simple interpretation of the difference and tendency structure functions: in regions where the logarithmic slope is between -1 and 0, it approximates the tendency structure function whereas in regions where the logarithmic slope is between 0 and 1, the calibrated Haar structure function approximates the difference structure function. For example, from the graph we can see that global scale temperature fluctuations decrease from $\approx 0.3 \text{ K}$ at monthly scales, to $\approx 0.2 \text{ K}$ at 10 yrs and then increase to $\approx 0.8 \text{ K}$ at $\approx 100 \text{ yrs}$. All of the numbers have obvious implications although note that they indicate the mean overall range of the fluctuations, so that for example the 0.8 K corresponds to $\pm 0.4 \text{ K}$ etc.

From fig. 5 we also see that the global surface temperatures separate into two regimes at about $\tau_c \approx 10 \text{ yrs}$, with negative and positive logarithmic slopes $= \xi(2)/ 2 \approx -0.1, 0.4$ for $\Delta t < \tau_c$ and $\Delta t > \tau_c$ respectively. Since $\beta = 1 + \xi(2)$ (eq. 11) we have $\beta \approx 0.8, 1.8$. We also analysed the first order structure function whose exponent $\xi(1) = H$; at these scales the intermittency ($K(2)$, eq. 4) $\approx 0.03$ so that $\xi(2) \approx 2H$ so that $H \approx -0.1, 0.4$ confirming that fluctuations decrease with scale in the low frequency weather regime but increase again at lower frequencies in the climate regime (more precise intermittency analyses are given in (Lovejoy and Schertzer, 2011)). Note that ignoring intermittency, the critical value of $\beta$ discriminating between growing and decreasing fluctuations (i.e. $H<0, H>0$) is $\beta = 1$. 


Fig. 5: A comparison of the different structure function analyses (root mean square, RMS) applied to the ensemble of three monthly surface series discussed in ch. 10 (NASA GISS, NOAA CDC, HADCRUT3), each globally and annually averaged, from 1881-2008 (1548 points each). The usual (difference, poor man’s) structure function is shown (orange, lower left), the tendency structure function (purple, lower right), the maximum of the two (“Hybrid”, thick, red), and the Haar in blue (as indicated); it has been increased by a factor $C = 10^{0.35} = 2.2$, the resulting RMS deviation with respect to the hybrid is ±14%. Reference slopes with exponents $\xi(2)/2 \approx 0.4, -0.1$ are also shown (black corresponding to spectral exponents $\beta = 1+\xi(2) = 1.8, 0.8$ respectively).

In terms of difference fluctuations, we can use the global root mean square $\langle \Delta T(\Delta t)^{1/2} \rangle$ annual structure functions (fitted for 129 yrs $> \Delta t >$ 10 yrs), obtaining $\langle \Delta T(\Delta t)^{1/2} \rangle \approx 0.08 \Delta t^{0.33}$ for the ensemble, in comparison, (Lovejoy and Schertzer, 1986) found the very similar $\langle \Delta T(\Delta t)^{1/2} \rangle \approx 0.077 \Delta t^{0.4}$ using northern hemisphere data (these correspond to $\beta_c = 1.66, 1.8$ respectively).

Before pursuing the Haar structure function let us briefly consider its sensitivity to nonscaling perturbations; i.e. to nonscaling external trends superposed on the data which break the overall scaling. Even when there is no particular reason to suspect such trends, the desire to filter them out is commonly invoked to justify the use of special wavelets – or nearly equivalently – of various orders of the MultiFractal Detrended Fluctuation Analysis technique (MFMDFA), (Kantelhardt et al., 2002). A simple way to produce a higher order Haar wavelet that eliminates polynomials of order $n$ is simply to iterate $(n+1)$ times the difference operator in eq. 14 for example, iterating three times yields the “quadratic Haar” fluctuation

$$(\Delta v(\Delta t))_{\text{quad}} = \frac{3}{\Delta t} (s(t + \Delta t) - 3s(t + \Delta t / 3) + 3s(t - \Delta t / 3) - s(t - \Delta t))$$

This fluctuation is sensitive to structures of size $\Delta t^1$ – and hence useful - over the range -1<H<2 and it is blind to polynomials of order 1 (lines). In comparison, the $n^{th}$ order DFA technique defines fluctuations using the RMS deviations of the summed series $s(t)$ from regressions of $n^{th}$ order polynomials so that
quadratic Haar fluctuations are nearly equivalent to the quadratic MFDFA RMS deviations (although these deviations are not strictly wavelets, note that the MFDFA uses a scaling function \( \approx \Delta v \Delta t \) hence with DFA exponent \( \alpha_{DFA}=1+H \)). Although at first sight the insensitivity of these higher order wavelets to external trends may seem advantageous, it should be recalled that on the one hand they only filter out polynomial trends - and not for example the more geophysically relevant periodic trends - while on the other hand, even for this, they are “overkill” since the trends they filter are filtered at all scales – not just the largest. Indeed, if one suspects the presence of external polynomial trends, it suffices to eliminate them over the whole series (i.e. at the largest scales), and then to analyse the resulting deviations using the Haar fluctuations.

Fig. 6 shows the usual (linear) Haar RMS structure function (eq. 14) compared to the quadratic Haar and quadratic MFDFA structure functions. It can be seen that the latter two are close to each other (after applying different calibration constants, see the figure caption), that the low and high frequency exponents are roughly the same. However, the transition point has shifted by nearly a factor of 3 so that overall they are rather different from the Haar structure function and it is clearly not possible to simultaneously calibrate the high and low frequency parts. The drawback with these higher order fluctuations is thus that we loose the simplicity of interpretation of the Haar wavelet and – unless \( H>1 \) - we obtain no clear advantage.

Fig. 6: The same temperature data as fig. 5; a comparison of the RMS Haar structure function (multiplied by \( 10^{0.35} = 2.2 \)), the RMS quadratic Haar (multiplied by \( 10^{0.15} = 1.4 \)) and the RMS quadratic MFDFA (multiplied by \( 10^{1.5} = 31.6 \)).

4. The transition from low frequency weather to the climate

4.1 Intermediate scale multiproxy series

In section 2 we discussed atmospheric variability over the frustratingly short instrumentally accessible range of time scales (roughly \( \Delta t<150 \text{ yrs} \)) and saw evidence that
weakly variable low frequency weather gives way to a new highly variable climate regime at a
scale $\tau_c$ somewhere in the range 10-30 years. In fig. 1 we already glimpsed the much longer 1-
100 kyr scales accessible primarily via ice core paleo temperatures (see also below); these
confirmed that - at least when averaged over the last 100 kyr or so - the climate does indeed have
a new scaling regime with fluctuations increasing rather than decreasing in amplitude with scale
($H > 0$).

Since the temporal resolution of the high resolution GRIP paleo temperatures was $\approx 5.2$
ys (and for the Vostok series, $\approx 100$ yrs) these paleo temperature resolutions don’t greatly
overlap the instrumental range; it is thus useful to consider other intermediates: the “multiproxy”
series that have been developed following (Mann et al., 1998). Another reason to use
intermediate scale data is because we are living in a climate epoch which is exceptional in both
its long and short term aspects. For example consider the long stretch of relatively mild and
stable conditions since the retreat of the last ice sheets about 11.5 kyr ago, the “Holocene”. This
epoch is at least somewhat exceptional: it has even been suggested that such stability is a
precondition for the invention of farming and thus for civilisation itself (Petit et al., 1999). It is
therefore possible that the paleoclimate statistics averaged over series 100 kyr or longer may not be
as pertinent as we would like for understanding the current epoch. Similarly, at the high
frequency end of the spectrum, there is the issue of “20th Century exceptionalism”, a
consequence of 20th century warming and the probability that at least some of it may be of
anthropogenic – not natural - origin. Since these affect a large part of the instrumental record, it
is problematic to use the latter as the basis for extrapolations to centennial and millennial scales.
In the following we try to assess both “exceptionalisms” in an attempt to understand the natural
variability in the last few centuries.

4.2 The Holocene exception: climate variability in time and in space

The high resolution GRIP core gives a striking example of the difference between the
Holocene and previous epochs in central Greenland (fig. 7). Even a cursory visual inspection of
the figure confirms the relative absence of low frequency variability in the current 10 kyr section
as compared to previous 10 kyr sections. To quantify this, we can turn to fig. 8 which compares
the RMS Haar structure functions for both GRIP (Arctic) and Vostok (Antarctic) cores for both
the Holocene 10 kyr section and, for the mean and spread of the eight earlier 10 kyr sections.
The GRIP Holocene curve is clearly exceptional, with the fluctuations decreasing with scale out
to $\tau_c \approx 2$ kys in scale and with $\xi(2)/2 \approx -0.3$. This implies a spectral exponent near the low
frequency weather value $\beta \approx 0.4$, although it seems that $\xi(2)/2 \approx 0.4 \ (\beta \approx 1.8)$ for larger $\Delta t$. The
main difference however is that $\tau_c$ is much larger than for the other series (see table 1 for
quantitative comparisons). The exceptionalism is quantified by noting that the corresponding
RMS fluctuation function ($S(\Delta t)$) is several standard deviations below the average of the
previous eight 10 kyr sections. In comparison (to the right in the figure) the Holocene period of
the Vostok core is also somewhat exceptional although less so: up to $\tau_c \approx 1$ kyr it has $\xi(2)/2 \approx -$
0.3 ($\beta \approx 0.4$) and it is more or less within one standard deviation limits of its mean although $\tau_c$ is
still large. Beyond scales of $\approx 1$ kyr its fluctuations start to increase; table 1 quantifies the
differences. We corroborated this conclusion by an analysis of the 2 kyr long (yearly resolution)
series from other (nearby) Greenland cores (as described in (Vinther et al., 2008)) where
(Blender et al., 2006) also obtained $\beta \approx 0.2-0.4$ and also obtained similar low $\beta$ estimates for the Greenland GRIP, GISP2 cores over the last 3 kyr.

While these analyses convincingly demonstrate that the Greenland Holocene was exceptionally stable, nevertheless their significance for the overall natural variations of northern hemisphere temperatures is not clear. For example, on the basis of paleo SST reconstructions just southeast of Greenland (Andersen et al., 2004) (also (Berner et al., 2008)) it was concluded that the latter region was on the contrary “highly unstable”. Using several ocean cores as proxies, Holocene SST reconstructions were produced which included a difference between maximum and minimum of roughly $6\,K$ and “typical variations” of $1-3\,K$. In comparison, from fig. 8 we see that the mean temperature fluctuation deduced from the GRIP core in the last 10 kyr is $\approx 0.2\,K$. However also from fig. 8 we see that the mean over the previous eight 10 kyr sections was $\approx 1-2\,K$, i.e. quite close to these paleo SST variations (and about amount expected in order to explain the glacial/interglacial temperature swings, see section 5). These paleo SST series thus underline the strong geographical climate variability effectively undermining the larger significance of the Greenland Holocene experience. At the same time they lend support to the application of standard statistical stationarity assumptions to the variability over longer periods (e.g. to the relevance of spectra and structure functions averaged over the whole cores).

Fig. 7: Four successive 10 kyr sections of the high resolution GRIP data, the most recent to the oldest from bottom to top. Each series is separated by 10 mils in the vertical for clarity (vertical units: mils). The bottom Holocene series is indeed relatively devoid of low frequency variability compared to the other 10 kyr sections, a fact confirmed by statistical analysis discussed in the text and fig. 8.
Fig. 8: This compares the RMS Haar structure function \( S(\Delta t) \) for both Vostok and GRIP high resolution cores (resolutions 5.2 and 50 years respectively over the last 90 kyrs). The Haar fluctuations were calibrated and are accurate to \( \pm 20\% \). For Vostok we used the Petit et al calibration, for GRIP, 0.5 K/mil. The series were broken into 10 kyr sections. The orange show the most recent of these (roughly the Holocene, top Vostok, bottom, GRIP) whereas the blue and green are the mean of the eight 10 – 90 kyr GRIP and Vostok \( S(\Delta t) \). The one standard deviation variations about the mean are indicated by dashed lines. Also shown are reference lines with slopes \( \xi(2)/2 = -0.3, 0.2, 0.4 \) corresponding to \( \beta = 0.4, 1.4, 1.8 \) respectively. Although the Holocene is exceptional for both series, for GRIP it is exceptional by many standard deviations. For the Holocene we can see that \( \tau_c \approx 1 \) kyr for Vostok, and \( \approx 2 \) kyr for GRIP, although for the previous 80 kyrs, we see that \( \tau_c \approx 100 \) yrs for both.

<table>
<thead>
<tr>
<th></th>
<th>( H )</th>
<th>( \beta )</th>
</tr>
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<tbody>
<tr>
<td><strong>Range of regressions</strong></td>
<td><strong>Holocene</strong></td>
<td><strong>10-90 kyrs</strong></td>
</tr>
<tr>
<td></td>
<td>100 yr &lt; ( \Delta t )</td>
<td>2 kyr &lt; ( \Delta t )</td>
</tr>
<tr>
<td></td>
<td>&lt; 2 kyr</td>
<td>&lt; 10 kyr</td>
</tr>
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<td><strong>GRIP</strong></td>
<td>-0.25</td>
<td>0.21</td>
</tr>
<tr>
<td><strong>Vostok</strong></td>
<td>-0.40</td>
<td>0.38</td>
</tr>
</tbody>
</table>

Table 1: This compares various paleo exponents estimated using the Haar structure function over successive 10 kyr periods. Vostok at 50 yr resolution, GRIP at 5.2 year resolution, all regressions over the scale ranges indicated. The Holocene is the most recent period (0-10 kyr). Note that while the Holocene exponents are estimates from individual series, the 10-90 kyr exponents are the means of the estimates from each 10 kyr section and (to the right), the
exponent of the ensemble mean of the latter. Note that the mean of the exponents is a bit below the exponent of the mean indicating that a few highly variable 10 kyr sections can strongly affect the ensemble averages. For the Holocene, the separate ranges < 2 kyr and Δt> 2 kyr were chosen because according to fig. 8, τ, ≈ 1 - 2 kyr.

4.3 Multiproxy temperature data, centennial scale variability and the 20th Century exception

The key to linking the long but geographically limited ice core series with the short but global scale instrumental series are the intermediate category of “multiproxy temperature reconstructions”. These series of northern hemisphere average temperatures pioneered by (Mann et al., 1998), (Mann et al., 1999) have the potential of capturing “multicentennial” variability over at least the (data rich) northern hemisphere. These series at typically annual resolutions combine a variety of different data types ranging from tree rings, ice cores, lake varves, boreholes, ice melt stratigraphy, pollen, Mg/Ca variation in shells, 18O in foraminifera, diatoms, stalactites (in caves), biota and historical records. In what follows, we analyze eight of the longest of these; see table 2 for some of statistical characteristics and descriptions.

Before reviewing the results, let us discuss some of the technical issues behind the continued development of new series. Consideration of the original series (Mann et al., 1998) (extended back to 1000 AD in (Mann et al., 1999)) illustrates both the technique and its attendant problems. The basic difficulty is in getting long series that are both temporally uniform and spatially representative. For example, the original six-century long multiproxy series presented in Mann et al 1998 has 112 indicators going back to 1820, 74 to 1700, 57 to 1600 and only 22 to 1400. Since only a small number of the series go back more than two or three centuries, the series’ “multicentennial” variability depends critically on how one takes into account the loss of data at longer and longer time intervals. When it first appeared, the Mann et al series created a sensation by depicting a “hockey-stick” shaped graph of temperature: with the fairly flat “handle” continuing from 1000 AD until a rapid 20thC increase. This lead to the famous conclusion - echoed in the 3rd IPCC report - that the 20thC was the warmest century of the millennium, that the 1990’s was the warmest decade and that 1998 was the warmest year. This success encouraged the development of new series using larger quantities of more geographically representative proxies (Jones et al., 1998), by the introduction new types of data (Crowley and Lowery, 2000), to the more intensive use of pure dendrochronology, (Briffa et al., 2001) or to improved methodologies (Esper et al., 2002).

However, the interest generated by reconstructions also attracted criticism, in particular, (McIntyre and McKitrick, 2003) pointed out several flaws in the (Mann et al., 1998) data collection and in the application of the principal component analysis technique which been had borrowed from econometrics. After correction, the same proxies yielded series with significantly larger low frequency variability and included the reappearance of the famous “medieval warming” period at around 1400 AD which had disappeared in the original. Later, an additional technical critique (McIntyre and McKitrick, 2005) underlined the sensitivity of the methodology to low frequency red noise variability present in the calibration data (the latter modelled this with exponentially correlated processes probably underestimating that which would have been found using long range correlated scaling processes). Other work in this period, notably by (von
Storch et al., 2004) using “pseudo-proxies” (i.e. the simulation of the whole calibration process with the help of GCM’s) similarly underlined the nontrivial issues involved in extrapolating proxy calibrations into the past.

Beyond the potential social and political implications of the debate, the scientific upshot was that increasing attention had to be paid to the preservation of the low frequencies. One way to do this is to use borehole data which – when combined with the use of the equation of heat diffusion has essentially no calibration issues whatsoever. (Huang, 2004) used 696 boreholes (only back to 1500 AD, the limit of this approach) to augment the original (Mann et al., 1998) proxies so as to obtain more realistic low frequency variability. Similarly, in order to give proper weight to proxies with decadal and lower resolutions, (especially lake and ocean sediments), (Moberg et al., 2005) used wavelets to separately calibrate the low and high frequency proxies. Once again the result was a series with increased low frequency variability. Finally, (Ljundqvist, 2010) used a more up to date, more diverse collection of proxies to produce a decadal resolution series going back to 1 AD. The low frequency variability of the new series was sufficiently large that it even included a 3rd century “Roman warm period” as the warmest century on record and permitted the conclusion that “the controversial question whether Medieval Warm Period peak temperatures exceeded present temperatures remains unanswered” (Ljundqvist, 2010).

With this context, let us quantitatively analyse the eight series cited above; we use the Haar structure function. We concentrate here on the period 1500-1979 because: a) it is common to all eight reconstructions, b) being relatively recent, it is more reliable (it has lower uncertainties) and c) it avoids the medieval warm period and thus the possibility that the low frequency variability is artificially augmented by the possibly unusual warming in the 1400’s. The result is shown in fig. 9 where we have averaged the structure functions into the five pre-2003 and three post-2003 reconstructions. Up to $\Delta t \approx 200$ years the basic shapes of the curves are quite similar to each other – and indeed to the surface temperature $S(\Delta t)$ curves back to 1881 (fig. 1). However quite noticeable for the pre 2003 constructions is the systematic drop in RMS fluctuations for $\Delta t \approx 200$ yrs which contrasts with their continued rise in the post 2003 reconstructions. This confirms the above analysis to the effect that the post 2003 analyses were more careful in their treatments of multicentennial variability. Table 2 gives a quantitative intercomparison of the various statistical parameters.

<table>
<thead>
<tr>
<th></th>
<th>$\beta$ (high freq ($4$-$10$ years)$^{-1}$)</th>
<th>$\beta$ (Lower freq than ($25$ years)$^{-1}$)</th>
<th>$H_{\text{high}}$ ($4$-$10$ years)</th>
<th>$H_{\text{low}}$ ($&gt;25$ years)</th>
<th>$C_1$</th>
</tr>
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<tbody>
<tr>
<td>(Jones et al., 1998)</td>
<td>0.52</td>
<td>0.99</td>
<td>-0.27</td>
<td>0.063</td>
<td>0.104</td>
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<td>(Mann et al., 1998), (Mann et al., 1999)</td>
<td>0.57</td>
<td>0.53</td>
<td>-0.22</td>
<td>-0.13</td>
<td>0.100</td>
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<tr>
<td>(Crowley and Lowery, 2000)</td>
<td>2.28</td>
<td>1.61</td>
<td>0.72</td>
<td>0.31</td>
<td>0.105</td>
</tr>
</tbody>
</table>
Table 2: A comparison of parameters estimated from the multiproxy data from 1500-1979 (480 years). The Ljundquist high frequency numbers are not given since the series has decadal resolution. Note that the $\beta$ for several of these series was estimated in (Rybski et al., 2006) but no distinction was made between low frequency weather and climate, the entire series were used to estimate single (hence generally lower) $\beta$’s.

<table>
<thead>
<tr>
<th>Source</th>
<th>$\beta$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
<th>$\beta_4$</th>
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<tr>
<td>(Briffa et al., 2001)</td>
<td>1.19</td>
<td>1.18</td>
<td>0.15</td>
<td>0.13</td>
<td>0.092</td>
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<td>(Esper et al., 2002)</td>
<td>0.88</td>
<td>1.36</td>
<td>0.01</td>
<td>0.22</td>
<td>0.092</td>
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<td>(Huang, 2004)</td>
<td>0.94</td>
<td>2.08</td>
<td>0.02</td>
<td>0.61</td>
<td>0.090</td>
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<td>(Moberg et al., 2005)</td>
<td>1.15</td>
<td>1.56</td>
<td>0.09</td>
<td>0.32</td>
<td>0.094</td>
</tr>
<tr>
<td>(Ljundqvist, 2010)</td>
<td>_</td>
<td>1.84</td>
<td>_</td>
<td>0.53</td>
<td>0.098</td>
</tr>
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</table>

Fig. 9: The RMS Haar fluctuation for the mean of the pre and post 2003 series from 1500 – 1979 (solid blue and purple respectively and excluding the Crowley series due to its poor resolution), along with the mean of the globally averaged monthly resolution surface series from fig. 5 in (solid) red. In order to assess the effect of the 20th C warming, the structure functions for the multiproxy data were recalculated from 1500 – 1900 only (dashed lines) and for the instrumental surface series with their linear trends from 1880 -2008 removed (the data from 1880-1899 is too short to yield a meaningful $S(\Delta t)$ estimate for the lower frequencies of interest). While the large $\Delta t$ variability is reduced a little, the basic power law trend is robust especially for the post 2003 reconstructions. Note that the decrease in $S(\Delta t)$ for the linearly detrended surface series over the last factor of 2 or so in lag $\Delta t$ is a pure artifact of the detrending. We may conclude that the low frequency rise is not an artifact of an external linear trend. Reference lines corresponding to $\beta = 0.8$ and 1.8 have been added.
Fig. 8 compares the mean multiproxies with the ensemble average of the instrumental global surface series. This confirms the basic behaviour: small $\Delta t$ scaling with $\xi(2)/2 = -0.1$ ($\beta = 0.8$) followed by large $\Delta t$ scaling with $\xi(2)/2 = 0.4$ ($\beta = 1.8$) is displayed by all the data, all the pre-2003 $S(\Delta t)$ functions drop off for $\Delta t \approx 200$ yrs. Notable are a) the transition scale in the global instrumental temperature at $\tau_c \approx 10$ yrs which is somewhat lower than that found in the reconstructions ($\tau_c \approx 40 - 100$ yrs) and, b) over the common low frequency part, that the amplitudes of the reconstruction RMS fluctuations are about a factor of two lower than for the global instrumental series. The reason for the amplitude difference is not at all clear since on the one hand, the monthly and annually averaged Haar structure functions of the instrumental series are very close to each other up to $\Delta t \approx 10$ yrs (the temporal resolution is not an issue) and similarly, the difference between the northern and the southern hemisphere instrumental $S(\Delta t)$ functions is much smaller than this (only about $\approx 15\%$).

To get another perspective on this low frequency variability we can compare the instrumental, and multiproxy structure functions with those from ice core paleo temperature discussed in more detail in the next section. In fig. 10, we have superposed the calibrated deuterium based RMS temperature fluctuations with RMS multiproxy and RMS surface series fluctuations (we return to this in section 5). We see that extrapolating the latter out to 30–50 kyr is quite compatible with the Vostok core with the “interglacial window” (i.e. the rough quasi period and amplitude corresponding to the interglacials). Although the Vostok $S(\Delta t)$ curve is from the entire 420 kyr record (not just the Holocene), this certainly makes it plausible that while the surface series appear to have realistic low frequency variability, that the variability of the reconstructions is too small (although the post-2003 reconstructions are indeed more realistic than the contrasting relative lack of variability in the pre-2003 reconstructions).

Fig. 10: This shows the RMS Haar fluctuations for the mean monthly, global surface series (orange, left, from fig. 5), the mean pre-2003 and mean post-2003 proxies (red, dark blue; bottom and middle left respectively, from fig. 9) as well as the mean Vostok $S(\Delta t)$ function over
the last 420 krys interpolated to 300 yr resolution and using the (Petit et al., 1999) calibration. Also shown is the “interglacial window”, the probable typical range of fluctuations and quasi periods of the glacial-interglacials. The “calibration” of the fluctuation amplitudes is accurate to ±25%.

4.4 20th Century Exceptionalism

Although the reconstructions and instrumental series qualitatively agree, their quantitative disagreement is large and requires explanation. In this section we consider whether this could be a consequence of the 20th century “exceptionalism” discussed earlier, the fact that irrespective of the cause – the 20th century is somewhat warmer than the 19th and earlier centuries. It has been recognized that this warming causes problems for the calibration of the proxies (e.g. (Ljundqvist, 2010)), and it will clearly contribute to the RMS multicennial variability in fig. 9. In order to demonstrate that the basic type of statistical variability is not an artifact of the inclusion of exceptional 20th C temperatures in fig. 9, we also show the corresponding Haar structure functions for the earlier period 1500-1900. Truncating the instrumental series at 1900 would result in a series only 20 years long, therefore the closest equivalent for the surface series was to remove overall linear trends, and then redo the analysis. As expected the figure shows that all the large Δt fluctuations are reduced, but that the basic scaling behaviours are apparently not affected. We conclude that both the type of variability as characterized by the scaling exponents and the transition scale τc are fairly robust – if difficult to accurately determine – they are not artifacts of external linear trends. The instrumental and reconstruction discrepancy in fig. 9 thus remains unexplained.

5. Temporal spectral scaling in the climate regime: 10 - 10^5 yrs

5.1 Review of literature

Thanks to several ambitious international projects, many ice cores exist, particularly from the Greenland and Antarctic ice sheets which provide surrogate temperatures based on δ18O or deuterium concentrations in the ice. The most famous cores are probably the GRIP (Greenland Ice core Project) and Vostok (Antarctica) cores, each of which are over 3 km long (limited by the underlying bedrock) and go back respectively 240 and 420 krys. Near the top of the cores, individual annual cycles can be discerned (in some cases going back over 10 krys); below that the shearing of ice layers and diffusion between the increasingly thin annual layers makes such direct dating impossible, and models of the ice flow and compression are required. Various “markers” (such as dust layers from volcanic eruptions) are also used to help fix the core chronologies.

A problem with the surrogates is their highly variable temporal resolutions combined with strong depth dependences. For example, (Witt and Schumann, 2005) used wavelets, (Davidsen and Griffin, 2010) used (monofractal) fractional Brownian Motion as a model, and (Karimova et al., 2007) used (mono) fractal interpolation to attempt to handle this, (Lovejoy and Schertzer, 2011) found that the temporal resolution itself has multifractal intermittency. The main consequence is that the intermittency of the interpolated surrogates is a bit too high but
that serious spectral biases are only present at scales of the order of the mean resolution or higher frequencies.

With these caveats table 3 summarizes some of the spectral scaling exponents, scaling ranges. It is interesting to note that the three main orbital (“Milankovitch”) forcings at 19 kyr, 23 kyr (precessional) and 41 kyr (obliquity) are indeed visible – but only barely - above the scaling “background”, see especially (Wunsch, 2003).

<table>
<thead>
<tr>
<th>Series</th>
<th>Authors</th>
<th>Series length (kyrs)</th>
<th>Resolution (yrs)</th>
<th>$\beta_c$</th>
</tr>
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<tbody>
<tr>
<td>Composite ice cores, instrumental</td>
<td>(Lovejoy and Schertzer, 1986)</td>
<td>Composite: Minutes to 10^6 years</td>
<td>1000</td>
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<tr>
<td>Composite (Vostok) (ice core, instrumental)</td>
<td>(Pelletier, 1998)</td>
<td>10^5 to 1000</td>
<td>0.1 to 500</td>
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<tr>
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<td>100</td>
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<td>Planktonic $\delta^{18}$O ODP677 (Panama basin)</td>
<td>(Wunsch, 2003)</td>
<td>1000</td>
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<td>$\delta^{18}$O from GRIP, Antarctica CO₂</td>
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<td>300</td>
<td>1.5</td>
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<td>(Ditlevsen et al., 1996)</td>
<td>91</td>
<td>5</td>
<td>1.6</td>
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<td>100</td>
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<td>0.1 to 10^7</td>
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<td>5</td>
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</tbody>
</table>

Table 3: An intercomparison of various estimates of the spectral exponents $\beta_c$ of the low frequency climate regime and scaling range limits. For series with resolution $\approx$ 100 yrs, the last three rows are for the (anomalous) Holocene only, see (Lovejoy and Schertzer, 2011).
5.2 A composite picture of atmospheric variability from 6 hours to $10^5$ years

To clarify our ideas about the variability, it is useful to combine data over huge ranges of scale into a single composite analysis (such as the spectra shown in fig. 1a, b) but using real space fluctuations rather than spectra. Some time ago, such a composite already clarified the following points: a) the distinction between the variability of regional and global scale temperatures b) that the global averages had particularly small transition scale $\tau_c$, c) that there was a scaling range for global averages between scales of about 3 years and 40 - 50 kys (where the variability apparently “saturates”) with a realistic exponent $\beta \approx 1.8$, d) that this scaling regime could potentially quantitatively explain the magnitudes of the temperature swings between interglacials (Lovejoy and Schertzer, 1986).

Similar scaling composites but in Fourier space were adopted by (Pelletier, 1998) and more recently, by (Huybers and Curry, 2006a) who made a more data intensive study of the scaling of many different types of paleotemperatures collectively spanning the range of about 1 month to nearly $10^6$ years. The results are qualitatively very similar - including the positions of the scale breaks; the main innovations are a) the increased precision on the $\beta$ estimates and b) the basic distinction made between continental and oceanic spectra including their exponents. We could also mention the composite of (Fraedrich et al., 2009) which is a modest adaptation of that of (Mitchell, 1976) although it does introduce a single scaling regime spanning only two orders of magnitude: from $\approx 3$ to $\approx 100$ yrs (with $\beta \approx 0.3$), although at lower frequencies the composite exhibits a decrease (rather than increase) in variability.

Fig. 11 shows an updated composite where we have combined the 20CR reanalysis spectra (both local, single grid point and global) with the GRIP 55 cm and GRIP high resolution spectra (both for the last 10 kyr and averaged over the last 90 kys), the three surface global temperature series. For reference, we have also included the 500 yr control run of the Institut Pierre Simon Laplace (IPSL) GCM used in the 3rd IPCC assessment report. We use difference structure functions so that the interpretation is particularly simple, although a consequence (see section 2) is that all the logarithmic slopes are $>0$. In order avoid this problem, compare this to the Haar structure functions, fig. 12.

Key points to note are a) the use of annually averaged instrumental data in fig. 11 (differences), but of daily data in fig. 12 (Haar), and b) the distinction made between globally and locally averaged quantities whose low frequency weather have different scaling exponents. Also shown is the interglacial window ($\Delta t$ is the half quasi- period, and for a white noise, $S$ is double the amplitude). The calibration of the paleotemperatures is thus constrained so that it goes through the window at large $\Delta t$ but joins up to the local instrumental $S(\Delta t)$ at small $\Delta t$. In addition, as discussed in section 4.2, since the last 10 kyr GRIP fluctuations are anomalously low (fig. 4 and see the nearly flat red curve compared with the full 91 kyr red curve); the calibration must be based on this flatter $S(\Delta t)$ (fig. 11). Starting at $\tau_c \approx 10$ yrs, one can plausibly extrapolate the global $S(\Delta t)$‘s using $H = 0.4$ ($\beta \approx 1.8$), all the way to the interglacial window (with nearly an identical $S(\Delta t)$ as in (Lovejoy and Schertzer, 1986)), although the northern hemisphere reconstructions don’t extrapolate as well, possibly because of their higher intermittency. The local temperatures extrapolate (starting at $\tau_c \approx 20$ yrs) with a lower exponent corresponding to $\beta$
≈1.4 (see fig. 1a) which is close to the other Greenland paleotemperature exponents (table 2 a) presumably reflecting the fact that the Antarctic temperatures are better surrogates for global rather than local temperatures, these exponents are all averages over spectra of series of ≈100 kyr or more in length. An interesting feature of the Haar structure function (fig. 12) is that it shows that local (grid scale) temperature fluctuations are roughly the same amplitude for monthly as for the much longer the glacial/interglacials periods. Not only can we make out the three scaling regimes discussed above, but for Δt> 100 kyr, we can start to discern a new low frequency climate regime. For more scaling paleoclimate analyses, including “paleo-cascades”; see (Lovejoy and Schertzer, 2011).

![Graph showing RMS structure function S(Δt) for different datasets over various time scales.](image)

Fig. 11: A comparison of the RMS structure function S(Δt) of the high resolution (5.2 yr) GRIP (red), IPSL (blue, 75°N and global), 20CR (gold, 75°N and global) mean surface series (purple), mean of the three post 2003 Northern hemisphere reconstructions (green) for globally averaged temperatures (bottom left set) and the mean at Greenland latitudes (upper set) all using fluctuations defined as differences (poor man’s wavelet) so that the vertical scale directly indicates typical changes in temperature. In addition, the GRIP data is divided into two groups: the Holocene (taken as the last 10 kyr, lower red) and the entire 91 kyr of the high resolution GRIP series (upper red). The GRIP δ¹⁸O data have been calibrated by lining up the Holocene structure function with the mean 75°N 20CR reanalysis structure function (corresponding to ≈0.65 K/mil). When this is done, the 20CR and surface mean global structure functions can be extrapolated with exponent H ≈0.4 (see the corresponding line) to the “interglacial window” (red box) corresponding to half pseudo-periods between 30 and 50 kyr with variations (= ±S/2) between ±2 and ±3 K. This line corresponds to spectral exponent β =1.8. Finally, we show a line with slope ξ(2)/2 =0.2 corresponding to the GRIP β =1.4 (see fig. 1 a, b); we can see that extrapolating it to 50 kyr explains the local temperature spectra quite well.
Fig. 12: This is the equivalent of fig. 11 except for the first order Haar structure function rather than the RMS difference structure function and including daily resolution 20CR data and monthly resolution surface temperatures. On the left top we show grid point scale (2°x2°) daily scale fluctuations for both 75°N and globally averaged along with reference slope $\xi(1) = H = -0.4$ (20CR, 700 mb). On the lower left, we see at daily resolution, the corresponding globally averaged structure function. Also shown are the average of the three in situ surface series (fig. 5) as well as the post 2003 multiproxy structure function (from fig. 9). At the right we show both the GRIP (55 cm resolution, calibrated with 0.5 K/mil) and the Vostok paleotemperature series. All the fluctuations have been multiplied by 2.2 so that the calibration scale in degrees $K$ is fairly accurate (compare with fig. 11). Also shown is the interglacial window and a reference line with slope -0.5 corresponding to Gaussian white noise.

6. Conclusions:

Just as the laws of continuum mechanics emerge from those of statistical mechanics when large enough numbers of particles are present, so do the laws of turbulence emerge at high enough Reynold’s numbers and at strong enough nonlinearity. However, the classical turbulence laws were constrained by strong assumptions of homogeneity and isotropy and could not cover wide scale ranges in the atmosphere. By generalizing them to take into account anisotropy (especially vertical stratification) and intermittency, their range of applicability is vastly increased. In the last five years - thanks in part to the ready availability of huge global scale data sets of all kinds - it has been possible to verify that these generalized emergent laws accurately hold up to planetary scales. For a recent review see (Lovejoy and Schertzer, 2010).

These “weather regime” laws show that the horizontal variability is fundamentally controlled by solar forcing via the energy flux. First principles calculations show that this
accurately accounts for the large scale winds and predicts a drastic “dimensional transition” at $\tau_w$.

≈10 days, the typical lifetime of planetary scale structures. Beyond this time scale, spatial
interactions are rapidly quenched so that the longer scales are driven by essentially temporal
degrees of freedom and the spectra of atmospheric fields display a shallow “spectral plateau”
with small logarithmic slope. By making a third generalization of the classical laws, the
statistical behaviour in this “low frequency weather” regime can be predicted. This qualitative
change in the statistics at $\tau_c$ is just as expected, it is not due to the action of new mechanisms of
internal variability nor to external climate forcing, it is apparently nothing more than low
frequency weather. Similarly, (forcing free) GCM control runs reproduce the same type of
variability out to their low frequency limits. The main complication is that - due to similar
effects from ocean turbulence whose corresponding outer time scale $\tau_o ≈ 1$ year – there is
enhanced intermittency up to that scale, with a slightly steeper “low frequency ocean weather”
regime beyond. Depending on the location and atmospheric variable of interest, this scaling
continues up to scales of $\approx 10 – 100$ years beyond which the type of variability drastically
changes; new mechanisms of internal variability or of external climate forcing must come into
play; the true climate regime begins. It is notable that if we consider pre-20th century northern
hemisphere temperatures, the result is qualitatively similar so that presumably anthropogenic
forcings are not responsible for this new regime.

In section 2, we discussed the weather and low frequency weather regimes using spectral
techniques. However in later sections we turned to fluctuation statistics. For these the key
parameter is the fluctuation exponent $H$ which determines the rate at which the mean fluctuations
increase ($H>0$) or decrease ($H<0$) with scale $\Delta t$. We underlined the particularly simple
interpretations afforded by the usual difference and tendency structure functions. However, the
Corresponding wavelets only usefully characterize the scaling of the fluctuations over narrow
ranges in the exponent $H$: the differences (0<$H<1$) and the tendency (-1<$H<0$), i.e. useful only
for fluctuations increasing in amplitude with scale or decreasing in amplitude with scale
respectively. Since both the weather and climate generally have $H>0$ and the low frequency
weather regime has $H<0$, we instead defined fluctuations using the Haar wavelet which is thus
useful over the entire range -1<$H<1$ and can also be “calibrated” to directly yield fluctuation
amplitudes and can be easily implemented numerically.

In order to evaluate the statistical variability of the atmosphere over as wide a range as
possible, we combined the Haar wavelet with temperature data from the 20th Century reanalysis
(1871-2008), (2°x2°, 6 hourly), three surface series (5°x5°, monthly), 8 intermediate length
resolution “multiproxy” series of the Northern hemisphere from 1500-1980 (yearly), and GRIP
and Vostok paleotemperatures at 5.2 and $\approx 100$ year resolutions over lengths 91 and 420 kyrs.
The basic findings were that the key transition scale $\tau_c$ from low frequency weather to climate,
was somewhat variable, depending on the field and geographical location. For example, for
surface global temperatures we found $\tau_c ≈ 10$ yrs whereas for the more reliable post 2003
northern hemisphere reconstructions $\tau_c ≈ 30$ yrs, for the Holocene GRIP (Greenland) core, $\tau_c ≈ 2$
kyrs, the Holocene Vostok (Antactica) core, $\tau_c ≈ 1$ kyrs, and the mean pre Holocene
paleotemperature value, $\tau_c ≈ 100$ yrs. We also found $H_{bw} ≈ -0.4$, -0.1 for local (e.g. 2°x2°
resolution) and global series respectively, and $H_c ≈ 0.4$ although the GRIP value was a little
lower these values correspond to $\beta_{bw} ≈ 0.2$, 0.8, 1.8 respectively (ignoring intermittency
corrections $K(2)$ – which ranged from $K_{bw}(2) ≈ 0.05$ to $K_c(2) ≈ 0.1$ a full characterization of the
intermittency i.e. $K(q)$ was been performed, but was not discussed here).
Although this basic overall three-regime scaling picture is 25 years old, much has changed to make it more convincing. Obviously, an important element is the improvement in the quantity and quality of the data, but we have also benefited from advances in nonlinear dynamics as well as in data analysis techniques. In combination, these advances make the three-regime scaling model a seductive framework for understanding atmospheric variability over huge ranges of space-time scales. It allows us to finally clarify the distinction between weather, the straightforward extension of the weather dynamics - without new elements - to low frequency weather and finally to the climate regime. It allows for objective definitions of the weather (scales \(<\tau_w\)), climate states (averages up to \(\tau_c\)), and hence of climate change (scales \(>\tau_c\)). This new understanding of atmospheric variability is essential in evaluating the realism of both atmospheric and climate models. In particular - since without special external forcing GCM’s only model low frequency weather - the question is posed as to what types of external forcing are required so that the GCM variability makes a transition to the climate regime with realistic scaling exponents and at a realistic time scales.

7. Acknowledgements

We thank P. Ditlevsen for providing the high resolution GRIP data, and A. Bunde for useful editorial and scientific comments.

8. References


T.J. Crowley and T.S. Lowery, How warm was the Medieval Warm period?, Ambio 29 (2000), pp. 51-54.


J.L. Goldman, The power spectrum in the atmosphere below macroscale. Institute of Desert Research, University of St. Thomas, Houston Texas (1968).


I. Van der Hoven, Power spectrum of horizontal wind speed in the frequency range from .0007 to 900 cycles per hour, *Journal of Meteorology* 14(1957), pp. 160-164.


