The Horizontal cascade structure of atmospheric fields determined from aircraft data

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Abstract:

Aircraft measurements of the power spectra of the horizontal wind field typically find a transition from $\approx k^{-5/3}$ to $\approx k^{-2.4}$ at scales somewhere around 40 km ($k$ is a wavenumber). In the usual interpretation this represents a transition between an isotropic 3-D ($k^{-5/3}$) and an isotropic 2-D ($k^{-3}$) turbulence; we have recently argued that the turbulence is so highly anisotropic that it has different exponents in the horizontal and vertical. When, coupled with gently sloping isobaric aircraft trajectories this predicts the break as a transition from a roughly horizontal spectrum at small scales to the spurious appearance of the vertical spectrum at large scales. If the atmosphere indeed has wide range horizontal scaling, then it is important to test out the multiplicative cascade models that predict its statistical behaviour. In this paper, we do this by analyzing wind, temperature, pressure and humidity data from the Winter Storm 2004 experiment using 24 aircraft legs each 1120 km long and at 280 m resolution. We analyse both the turbulent fluxes and the fluctuations showing that in spite of the non-flat trajectories that there is good evidence of roughly planetary scale multiplicative cascades. By carefully determining the scale-by-scale effects of intermittency on the aircraft altitude and measurements we estimate the corresponding scaling exponents. We argue that our results should finally permit the emergence of a long needed consensus about the basic scale-by-scale statistical properties of the atmosphere. They also point to the urgent need to development anisotropic scaling models of turbulence.
1. Introduction:

1.1 Reinterpreting the statistics of the horizontal wind:

Data analysis requires a theoretical framework in which the physical quantities can be defined and understood. In the case of aircraft measurements of atmospheric turbulence, it is usual to interpret the measurements in an isotropic or quasi isotropic framework in which the basic exponents are the same in the horizontal and vertical directions. However, starting in the 1980’s studies of the vertical structure of the atmosphere have almost invariably concluded that the horizontal wind is scaling in the vertical, but with a different exponent than in the horizontal. This empirically observed scaling stratification has thus been interpreted either with quasi-linear gravity wave models [Van Zandt, 1982], [Gardner, 1994], [Gardner et al., 1993], [Dewan, 1997], [Koch et al., 2005] or the 23/9D strongly turbulent model, [Schertzer and Lovejoy, 1985], [Lovejoy et al., 2008b].

In the last few years, these results have prompted the examination of the consequences of such anisotropic but scaling turbulence for the interpretation of aircraft data; both for stratospheric aircraft following isomachs [Lovejoy et al., 2004] and for tropospheric aircraft following isobars (or other gradually sloping trajectories) [Lovejoy et al., 2009c]. The perhaps surprising conclusions were that they are best explained by a single wide range anisotropic scaling regime. A corollary is that the usual interpretation - that the atmosphere has two (or more) isotropic scaling regimes (e.g. small scale, 3D isotropic and large scale, 2-D isotropic) is untenable. The key finding was that when
measuring the wind field, rather than detecting successively two different isotropic
turbulence exponents, the aircraft first detects at small scales the correct horizontal
exponent, and then - at larger scales where the departure of the aircraft from a roughly
flat to a sloping trajectory becomes important - it spuriously measures the different
vertical exponent. In terms of traditional spectral exponents $\beta$, (i.e. the spectra are of the
form $E(k) \approx k^\beta$ where $k$ is a wavenumber), the horizontal has $\beta \approx 5/3$ (near the
Kolmogorov value) while the vertical has $\beta \approx 2.4$. More precisely - as accurately
estimated by drop sondes [Lovejoy et al., 2007] - $\beta$ increases roughly monotonically
from about 2.15±0.04 near the surface to 2.51±0.04 at 10 km altitude, see fig. 1c below.
The near surface value is – to within intermittency corrections – that predicted for
buoyancy driven turbulence by [Bolgiano, 1959], [Obukhov, 1959] and the error bars
indicate the spread between the mean sonde to sonde exponent and the overall regression
exponent using 235 sondes.

As pointed out in [Lilley et al., 2008] and especially [Lovejoy et al., 2009c], the
slightly larger upper troposphere values 2.4 – 2.5 are almost exactly the same as the
values of the large scale exponents found in all the major aircraft campaigns including
GASP [Nastrom and Gage, 1985], MOZAIC [Cho and Lindborg, 2001] and others [Gao
and Meriwether, 1998], [Bacmeister et al., 1996]. Fig. 1a adapted from [Skamarock,
2004] neatly compares on a single graph, the two largest aircraft campaigns to date:
GASP and MOZAIC. It can be seen that – as predicted - at scales of about $\approx 100$ km
that the exponents shift from the small scale Kolmogorov value $\beta \approx 5/3$ to the vertical
value $\beta \approx 2.4$. The spectra in fig. 1a are extremely close to those found in [Lovejoy et
al., 2009c] using the same data as below (see fig. 4c), except that the transition scale
(which varies greatly from trajectory to trajectory) is on average $\approx 40 \text{ km}$ rather than $\approx 100 - 200 \text{ km}$. We should note that the persistence of a $k^{-5/3}$ regime to such large scales is a fundamental problem for the standard 2D/3D model since at those scales the turbulence cannot be isotropic. Various speculative mechanisms to explain it include the description “escaped” 3D energy transformed to quasi 2D stratified turbulence [Lilly, 1989], “squeezed 3D isotropic turbulence” [Högström et al., 1999]) and “upscale quasi-2D nonlinear KE cascades from smaller scales where the KE is stirred by moist convection” [Takayashi et al., 2006]. Other unsatisfactory features are the unknown flux sinks in the 2-D/3-D transition region, an unknown large scale energy flux dissipation mechanism (surface drag?), and speculative energy and enstrophy flux sources at $\approx 2000 \text{ km}$. 
Fig. 1a: An intercomparison of the GASP and MOZAIC spectra from commercial aircraft flying on isobars, adapted from [Skamarock, 2004]. The red lines show the behaviour predicted if the atmosphere has a perfect $k^{5/3}$ horizontal spectrum but estimated from an aircraft following roughly horizontal trajectories until about 100 km and then following gradually sloping trajectories (either on isobars or gradual changes in altitude due to fuel consumption).

These empirical results can be explained if atmospheric dynamics are scaling but strongly anisotropic i.e. with different exponents in the horizontal and vertical directions. To put this anisotropic model in perspective, recall that the classical model of atmospheric motions first postulates isotropy (in either two or three dimensions) and only then scaling. Since the troposphere is only $\approx 10$ km thick large structures must be “flattened”; that such isotropic models therefore require at least two scaling regimes, the usual ones being dominated by energy flux (leading to a small scale Kolmogorov, $\beta = 5/3$ regime) and a large scale ($\beta = 3$) enstrophy flux dominated regime (actually such models require a third even larger scale $\beta = 5/3$ inverse energy flux cascade regime, but this is rarely discussed). This classical model has been dominant ever since [Charney, 1971] extended [Kraichnan, 1967]'s pure hydrodynamic two dimensional turbulence model to quasi geostrophic turbulence. In comparison, the anisotropic scaling model implies that the stratification increases in a power law manner so that structures 20,000 km long in the horizontal may only be 10 km thick in the vertical. It turns out that all that is required to obtain a spurious $k^{-2.4}$ regime is a small mean aircraft slope with respect to the horizontal – e.g. due to a slow rise due to fuel consumption, or along an isobaric surface. If the isobars were sufficiently rough, then following an isobar might lead to yet another behaviour. However, isobars and geopotentials are very smooth. More precisely, if their spectral slopes $\beta$ are greater than 3 (as in e.g. [Trenberth and
Then their r.m.s. variability is due to low not high wavenumbers, so
\( \beta > 3 \) is at least a rough criterion for “smooth enough” suggesting that sloping isobars and
straight sloping trajectories will yield the same behaviour.

As model resolutions increased, the predictions of this standard 2D/3D model
could be directly tested numerically – at least at the large scales. The first tests, [Chen
and Wiin-Nielsen, 1978], [Boer and Shepherd, 1983] had very small ranges of scales and
are widely cited as giving support to 2-D turbulence, especially its prediction of a \( k^3 \)
regime. However, given the long history of “shoe-horning” atmospheric spectra into a
\( k^3 \) mould, re-examination of their original results is salutary; see the [Boer and
Shepherd, 1983] key result in fig. 1b which shows the enstrophy spectrum (= \( k^2 E(k) = k^2 \)
\( \beta \)). We see that although for \( k > 5 \) the vertically integrated and transient spectra are very
close to \( \beta = 2.4 \), the stationary spectrum is close to \( \beta \approx 4 \), with no indication of a \( \beta \approx 3 \)
regime. This prompted the following prescient comment: “For the purposes of
comparison with theory, the spectral slopes obtained form the data are somewhat
shallower than the values of -3 suggested by simple theory. It must be emphasized
however, the enstrophy containing inertial subrange is not really a prediction for the
atmosphere but is a possible solution to the spectral equation in an unforced subrange
which may or may not have some correspondence to the situation in the real
atmosphere”, [Boer and Shepherd, 1983]. They continue: “Consequently, the fact that
the spectra obey power laws at all may be considered to be a striking, although by now
well known feature of the atmosphere”. This primacy of the scale invariance symmetry
– albeit a strongly anisotropic one – is the basis of the alternative discussed here. Later,
[Strauss and Ditlevsen, 1999], applied the same [Boer and Shepherd, 1983] 2D analysis
framework to much higher resolution ERA 40 reanalyses also finding “roughly $\beta \approx 2.5$-2.7… this slope is significantly different than the classical turbulence theory prediction of 3” a fact which they partially attribute to a “lack of enstrophy cascade”.

Fig. 1b: The enstrophy spectrum ($= n^2 E(n)$) where $E(n)$ is the wind spectrum and $n$ is the principal spherical harmonic wavenumber), adapted from Boer and Shepherd 1983. The three curves are from January data; the solid line is for the vertically integrated atmosphere, the lines indicated stationary, (spatial spectrum of the monthly average) and the transient is the deviation from the monthly average. Over the range $n \approx 5$ to 30 (700-4000 km) the exponents of the spectra of the transient and vertically averaged atmosphere are extremely close to the vertical value $\beta \approx 2.4$, but the stationary spectrum exponent is $\beta \approx 4$. No $\beta \approx 3$ regime is observed.
Fig. 1c: This shows the isotropic spectrum of zonal component of the wind at 200, 300, 400, … 1000 mb from the ECMWF interim reanalysis for Jan. 2006 averaged between ±45° latitude. The straight lines are not regressions, rather they have the slopes of the horizontal wind in the vertical direction as estimated by drop sondes in Lovejoy et al 2007. It can be seen that the isobaric velocity spectra have exponents close to the vertical values (especially at the data rich lower levels).

Today, we can revisit wind spectra using the state-of-the-art successor to the [Strauss and Ditlevsen, 1999] data, the ECMWF interim reanalysis and calculate the spectrum directly without Strauss and Ditlevsen's, complex 2D preprocessing. Fig. 1c shows the isotropic spectrum of the zonal wind at each tropospheric 100 mb level, compensated by the average $k^{-2.4}$ behaviour so as to accentuate the small deviations. Also shown in the figure are straight reference lines. These are not regressions but rather the predictions of the model presented here: the slopes are those empirically estimated in the vertical direction from drop sondes [Lovejoy et al., 2007]. Regressions on the reanalysis
spectra from $k = 2$ to $k = 30$ give $\beta$ differing by less than 0.05 throughout the data rich lower 4 km, rising to only 0.2 at 10 km ($\approx 200$ mb). These small differences could easily be the consequence of either intermittent aircraft and/or sonde motion, see below. For reference, the exact values at 100 mb levels are: (1000, 2.15, 2.16), (900, 2.29, 2.26), (800, 2.36, 2.32), (700, 2.38, 2.40), (600, 2.40, 2.48), (500, 2.43, 2.53), (400, 2.47, 2.62), (300, 2.52, 2.68), (200, 2.45, 2.82) for (pressure in mb, drop sonde $\beta$, horizontal reanalysis $\beta$).

A possible explanation for the difficulties with the standard 2-D/3D picture comes from work by [Bartello, 1995], [Ngan et al., 2004]. These authors pointed out that the standard model’s small scale 3-D turbulence could destabilize any large scale 2-D turbulence. This apparent internal contradiction offers a possible explanation for the failure to find either a $k^3$ regime or enstrophy cascade [Strauss and Ditlevsen, 1999] in the reanalyses. Such instability may also explain why some models such as the ECMWF forecast model apparently fail to yield $k^{5/3}$ regimes [Palmer, 2001] yet at the same time, Princeton’s Geophysical Fluid Dynamics Laboratory SKYHI model apparently are able to yield both $k^{5/3}$ and $k^3$ regimes; see fig. 1d from [Takayashi et al., 2006] and see [Hamilton et al., 2008]. Similar claims for simultaneous $k^{5/3}$ and $k^3$ regimes have been made for the regional WRF model [Skamarock, 2004], but even if the claims are valid, the $k^3$ regime is over a very narrow range and in addition, the purported $k^{5/3}$ range is much too shallow: fig. 1e shows that in fact a single $k^{-2.4}$ regime works extremely well over the whole range.
Fig. 1d: This shows the comparison of a large (T639) simulation on the Earth Simulator (zonal wind, at 45°N, 200 mb, forecasts for days 6-15 shown in red) with a replotting of the GASP aircraft spectra (cyan, crosses), adapted from [Takayashi et al., 2006]. The solid black lines show the predicted isobaric and horizontal spectra have been added for reference. It can be seen that while overall the model reproduces the GASP spectra reasonably well down to the model dissipation scales, it is striking that the largest deviations of the model from the empirical spectra are precisely in the region 400 - 3000 km (between the arrows) where the isobaric $k^{-2.4}$ spectrum is very accurate.
Fig. 1e: This shows sample spectra from WRF forecasts of zonal wind averaged over the isobaric surfaces covering roughly the range 3-9 km in altitude, adapted from [Skamarock, 2004]. The claimed “clear $k^3$ regime” for the solid (oceanic) spectrum spans a range of factor 2 - 3 at the relatively unreliable extreme low wavenumbers (between the arrows, upper left). Except for the extremes, the spectra again follow the isobaric predictions $k^{2.4}$ (red) very well over most of the range.

Ironically, the ability of the SKYHI or possibly the WRF models to yield two scaling regimes may not be an indication of their realism. For example, returning to fig. 1d which compares the Takayashi et al earth simulator results (6-15 day forecast) with the GASP data, we see that the scales with the largest deviations between the model and the empirical spectra (the range $\approx400 – 3000$ km, between the arrows) are precisely those which are supposedly explained by the existence of the two regimes! On the other hand, cascade analyses of the type described below - which neither make a priori assumptions
about the physical nature of the cascade quantity nor of its isotropy or anisotropy - show
that both the ERA 40 reanalyses in either space, [Stolle et al., 2009] or in time [Stolle,
2009], or meteorological models of the atmosphere (the GFS and GEM models) have
nearly perfect scaling cascade structures over almost their entire range of spatial scales;
see e.g. fig. 1f or the numerous figures in the above references (the deviations from
multiplicative cascades are between ≈ 0.5 – 1% up to 5000 km in scale). As argued in
[Schertzer and Lovejoy, 1987] or in the review [Lovejoy and Schertzer, 2009] the reason
that this is possible (even expected), is that the models - and apparently the atmosphere -
are both anisotropic but scale invariant over huge ranges. Note that boundary conditions
such as the topography as well as the short and long wave atmospheric forcings also
display scaling cascade structures over most of their observed ranges so that they are not
expected to break the dynamical scaling [Gagnon et al., 2006], [Lovejoy, 2001],
[Lovejoy et al., 2009a].
Fig. 1f: This shows the spatial scaling of the energy flux ($\varepsilon$) estimated from the Laplacian of the zonal wind at 1000 mb, for the GEM model at $t = 0$ (reproduced from [Stolle et al., 2009], for the details of the technique, see section 2 below). The statistical moments $M_q$ are shown for $q = 0.0$ to $2.9$ ($q > 1.0$: $\log_{10} M_q > 0$, monotonically increasing; $q < 1.0$: $\log_{10} M_q < 0$) in steps of $0.1$, $\lambda = L_{\text{earth}}/L$, $L_{\text{earth}} = 20,000$ km. The $q$ $\text{th}$-moment colours vary from $q = 0$ (reddish-orange) to $q = 2.9$ (red). The converging straight lines show that the statistics up to order 2.9 are well approximated by a multiplicative cascade starting at the about 10,000 km. The results for the GFS model are almost identical as are the forecast fields at $t = 48, 144$ hours. For scales $< 5000$ km the deviations from the straight lines are $\approx \pm 0.6\%$. This may be compared with the empirical aircraft results derived below (fig. 2b).

1.2 Characterizing the horizontal scaling of atmospheric fields:

If the atmosphere is anisotropic and scaling over wide ranges, then it is fundamentally important to characterize the anisotropic scaling regimes for as many of the fields as possible, i.e. to determine the type of scaling (including the determination of the basic turbulent parameters), as well as the limits to the scaling – the inner and outer
scales. The goal of this paper is therefore to go a step beyond the reinterpretation outlined above to see how the data can be quantitatively used to characterize atmospheric turbulence while attempting to avoid spurious aircraft induced effects.

The processes which account for the wide range anisotropic scaling of the fields are the anisotropic multiplicative cascade processes, i.e. anisotropic extensions of the explicit phenomenological cascade models that were developed through the 1960’s, 70’s and 80’s ([Novikov and Stewart, 1964], [Yaglom, 1966], [Mandelbrot, 1974], [Schertzer and Lovejoy, 1987]). In recent papers, we have investigated the vertical cascade structure using drop sondes [Lovejoy et al., 2009b], the horizontal cascade structure using satellite data [Lovejoy et al., 2009a] and vertical cross-sections using lidar backscatter data ([Lilley et al., 2004], [Lilley et al., 2008]). In addition, meteorological reanalyses as well as numerical meteorological models [Stolle et al., 2009] have recently been shown with high accuracy to have multiplicative cascade structures over almost their entire spatial ranges and up to about 10 days in the time domain. What is missing from a fairly complete spatial characterization of atmospheric cascades is the direct verification of the multiplicative cascade structure in the horizontal on the standard meteorological fields including the horizontal wind, the temperature, humidity, potential temperature; this is our task here.

This paper is structured as follows. In section 2 we review the data and discuss some of its limitations, including the connection between the wind, pressure and altitude fields, we introduce the cascade formalism. In section 3, we discuss the basic flux and fluctuation analyses of the wind, temperature, potential temperature, humidity fields, in section 4 we give a refined treatment attempting to determine the best regime for
parameter estimation, and to get the best parameter values. In section 5 we discuss outstanding issues and conclude.

2. The data, its Intermittency:

The data analysed were from the Winter Storm 2004 experiment and involved 10 aircraft flights over a roughly a 2 week period over the northern Pacific. Each flight dropped 20 - 30 drop sondes which were analyzed to determine the vertical cascade structure [Lovejoy et al., 2009b]. For air traffic reasons, the Gulfstream 4 plane flew along either the 162, 178, or 196 mb isobars, (to within ±0.11mb i.e. the pressure level was ≈ constant to within ±0.068%), it has a radar altimeter which is reliable over the ocean, but which is only used to anchor the GPS estimates, these geometric altitudes were used here. Each had one or more roughly constant straight and constant altitude legs more than four hundred kilometers long between 11.9, 13.7 km altitude. The data were sampled every 1 s and the mean horizontal aircraft speed with respect to the ground was 280 m/s. In addition, we checked that the distance covered on the ground between measurements was constant to within ±2% so that the horizontal velocity was nearly constant (in addition, using interpolation, we repeated the key analyses using the actual ground distance rather than the elapsed time and found only very small differences). This is the same data set analyzed in [Lovejoy et al., 2009c] where it is described in more detail.

The basic scale by scale relations between the trajectory and the fields was investigated using both spectral and cross-spectral analysis in [Lovejoy et al., 2009c]. Fig. 2a shows an extract of the latter results for simplicity showing only the results for the
longitudinal (the along trajectory) component of the wind. We calculated the cross-
spectrum which is a kind of wavenumber by wavenumber cross-correlation coefficient.  
However, unlike the usual cross-correlation, it is complex-valued hence it is usual to  
introduce the modulus – called the “coherency” (C) - and argument, the “phase” (θ; see  
e.g. [Landahl and Mollo-Christensen, 1986]). An important difference between the  
coherency and a cross-correlation is that C is always positive; in fig. 2a, C ≈ 0.2 implies  
a statistically significant relation. On the other hand, a positive phase in fig. 2a implies  
that the wind leads (pressure or altitude), a negative phase, that it lags behind. From the  
figure we see that between about 4 and 40 km, the altitude leads the wind but the pressure  
lags behind: the situation is reversed at larger scales (smaller wavenumbers). The direct  
interpretation is that for the higher wavenumbers \(((4 \text{ km})^{-1} > k > (40 \text{ km})^{-1})\), corresponding  
to time scales of 10 - 150 s) the aircraft autopilot and inertia cause the change in altitude  
with the pressure then following the altitude. For the smaller wavenumbers \((k < (40 \text{ km})^{-1})\), the situation is reversed with the pressure changes leading (presumably causing)  
the change in wind and altitude; this is presumably the regime where the aircraft tightly  
follows the isobars.
Fig. 2a: Coherencies ($C$, right axis) and phases ($\theta$ in radians, left axis) of the longitudinal wind with pressure (blue) and altitude (red). Solids are coherencies; those greater than $\approx 0.2$ are statistically significant, they are highly significant over most of the range. Thick dashed lines are phases, thin purple dashed lines are the one standard deviation confidence intervals for the phases (they increase at low wavenumbers due to the smaller number of samples). A positive phase means that the wind leads (pressure or altitude), a negative phase, that it lags behind.

The coherency and phase analysis suggests that the main effect of the trajectory fluctuations is on the wind field and that its influence will be smaller for the other atmospheric fields (we see this below). Going beyond phases and coherencies, let’s consider the detailed statistics of the trajectories. For scales less than about 4 km, the aircraft inertia smoothes them out; the main variations are smooth and are associated with
various roll modes and the pilot/autopilot response. However, at scales from about 4 to 40 - 100 km (and from ≈3 to about 300 km for the stratospheric ER-2 trajectories along isomachs), the trajectory is fractal with the altitude lagging behind the (horizontal) wind fluctuations. Finally at scales ≥≈ 40 km the aircraft follows the isobars quite closely so that the wind lags behind the pressure. These conclusions were reached by considering the average for 24 aircraft legs flying between roughly 11 and 13 km, each 1120 km long as well as through a leg by leg analysis which showed considerable variability in the scale at which these transitions occurred. According to this analysis we may anticipate that there will be a strong effect of the variability (indeed, intermittency) of the aircraft altitude on the measurements – at least for scales smaller than about 40 km where the wind leads the altitude, imposing its strong intermittency on the aircraft.

In order to demonstrate this and to quantify the intermittency of the trajectory we performed a standard multifractal analysis. Since we will perform similar analyses on the meteorological fields we give a general explanation of the method, using the example of the familiar Kolmogorov law of turbulence.

Consider wind fluctuations \( \Delta v \) over distances \( \Delta x \) with underlying turbulent energy flux \( \phi \); we therefore have:

\[
\Delta v = \phi \Delta x^H
\]  

(1)

The form 1, is a generic relation between a turbulent fluctuation \( \Delta v \) and turbulent flux \( \phi \) (at resolution \( \Delta x \)), the Kolmogorov law is the special case where \( \phi = e^{1/3} \) and \( H = 1/3 \).

Strictly speaking, the equality in eq. 1 is in the sense of probability distributions, i.e. the distribution of the random fluctuation \( \Delta v \) is the same as that of \( \phi \Delta x^H \). In the following,
this distinction is not important since we use $\Delta v/\Delta x$ to investigate the statistical properties of $\phi$, and equality in this weaker probabilistic sense is sufficient. In the data analyzed below, we generally took $\Delta v(\Delta x) = |v(x + \Delta x) - v(x)|$ but other definitions – such as the second finite difference rather than the first – or equivalently wavelets may be used.

If over a range of scales, $\phi$ is the result of a multiplicative cascade process, then the normalized flux $\phi^*_\lambda = \phi_\lambda / \langle \phi \rangle$ obeys the following statistics:

$$M_q = \left( \frac{\lambda}{\lambda_{\text{eff}}} \right)^{K(q)}; \quad \lambda = \frac{L_{\text{ref}}}{L}; \quad \lambda_{\text{eff}} = \frac{L_{\text{ref}}}{L_{\text{eff}}}$$

$$K(q) \quad \text{(2)}$$

Where $M_q = \langle \phi^{*q}_\lambda \rangle = \langle \phi^{q}_\lambda \rangle / \langle \phi \rangle^q$ is the normalized $q^{\text{th}}$ moment at resolution $L$ (scale ratio $\lambda$), $L_{\text{ref}}$ is a convenient reference scale (taken below as the largest great circle distance on the earth, $L_{\text{ref}} = 20,000$ km), $L_{\text{eff}}$ is the “effective” scale at which the cascade begins, and $K(q)$ is the scaling exponent characterizing the intermittency. Note that there is no need for a subscript on $\langle \phi \rangle$ since the ensemble flux is a climatological value independent of the resolution $\lambda$ (i.e. $K(q = 1) = 0$ which follows since for $q = 1$ the operations of spatial and ensemble averaging commute). In order to test eq. 2 on the data, it is sufficient to use the (absolute) fluctuations at the smallest available scales $l$; (i.e. take $l = \Delta x$, corresponding to the large scale ratio $\Lambda$); and then to estimate the normalized flux as:

$$\frac{\phi_\lambda}{\langle \phi_\lambda \rangle} = \frac{\Delta v(l)}{\langle \Delta v(l) \rangle}; \quad \Lambda = \frac{L}{l} \quad \text{(3)}$$
i.e. it is not necessary to know $H$ or even the physical nature of the flux $\phi$ so that our results are independent of any specific theory of turbulence. The fluctuations can be estimated either by differences (when $0 < H < 1$) or (equivalently) by wavelets. The normalized flux at lower resolution $\lambda < \Lambda$ is then obtained by straightforward spatial averaging of the fine scale ($\Lambda$) resolution normalized fluxes. We do not attempt to determine the direction (i.e. from large to small or small to large) of the cascade. Note that we follow standard conventions using $\Delta x$ for the length scale of fluctuations and $l$ and $L$ for the resolutions of fluxes; $\Lambda, \lambda$ (both $\geq 1$) are dimensionless scale ratios, largest scale to inner resolution scale ($\Lambda$ being the smallest available, $\lambda$ can vary in the range $1 \leq \lambda \leq \Lambda$).

When this method is applied to the aircraft altitude, we obtain figure 2b which shows that eq. 2 is well verified with lines converging to $\log_{10} \lambda_{eff} \approx 2.9$ corresponding to $L_{eff} \approx 30$ km. This shows that for scales $\ll 30$ km, the altitude has very strong intermittency which is of the type theoretically predicted for turbulent cascades. This result is compatible with the phase and coherence analysis summarized above to the effect that for scales $< \approx 40$ km that the phase of the wind leads the phase of the altitude. The simplest explanation is that the altitude is intermittent because it varies in response to the strong wind intermittency. At the larger scales the phase changes sign and the pressure leads the wind so that (due to the autopilot) the aircraft starts to closely follow the isobars which are much smoother (the response time of the autopilot is much shorter than the 2-3 minutes needed to cover $\approx 40$ km but apparently, the turbulence is too strong at smaller scales).
We now seek to quantify the accuracy with which the altitude intermittency follows the cascade form. First, we may use the logarithmic slopes of $M_q$ to estimate $K(q)$, and then use a parametric form of $K(q)$ in order to reduce it to a manageable (finite) number of parameters. This is most conveniently done by exploiting the existence of stable, attractive cascade processes - the result of a kind of multiplicative central limit theorem - which gives rise to “universal multifractals” [Schertzer and Lovejoy, 1987], [Schertzer and Lovejoy, 1997]. For universal multifractals $K(q)$ has of the form:

$$K(q) = \frac{C_1}{\alpha - 1}(q^\alpha - q); \quad q \geq 0; \quad 0 \leq \alpha \leq 2$$  \hspace{1cm} (4)

The parameter $C_1$ ($0 < C_1 < D$) characterizes the intermittency of the mean: $C_1 = K'(1)$, it thus characterizes the intermittency of the mean field ($D$ is the dimension of the observing space, here $D = 1$). The second parameter is the multifractal index $\alpha$ which characterises the degree of multifractality, it quantifies how rapidly (with intensity, with $q$), the statistics deviate from the monofractal case $\alpha = 0$). In table 1 we give the estimates of the parameters for the altitude and various other fields. We return in section 4 to the parameters attempting refined estimates and also comparing the values with those of reanalyses and meteorological models.

To quantify the accuracy, we may characterize the deviations by the mean absolute residuals for the statistical moments $M_q$ of order $q$ from 0 to 2 for all points between the scale of the grid and the scales in the figure (about 100 times larger):

$$\Delta = \left| \log_{10}(M_q) - K(q) \log_{10}(\lambda / \lambda_{\text{eff}}) \right|$$  \hspace{1cm} (5)
To convert $\Delta$ to a percent deviation, use $\delta = 100(10^\Delta - 1)$, which we find for the altitude (fitting between 1 km and 20 km) is $\approx \pm 2.6\%$ (see table 1 for the mean $\bar{\delta}$ averaged over the $q \leq 2$).

Fig. 2 b: The normalized moments $q = 0.2, 0.4, \ldots, 3$, for aircraft altitude, $z$; $\log_{10}\lambda$ corresponds to 20,000 km ($= L_{red}$).
3. Scaling analyses:

3.1 Trace moments and the cascade structure of the fields:

Based on the analysis of the aircraft altitude and cross-spectra, we have argued that the aircraft trajectory fluctuations and their coupling with the fields must be taken into account for quantitative analysis of the statistics. However, before we attempt a more refined analysis, we present the basic cascade (flux) and then fluctuation analyses.

As indicated in section 2 (c.f. eq. 3), the dimensionless, normalized flux is estimated using absolute fluctuations at the smallest scales. In the case of the horizontal wind in the $\approx k^{-5/3}$ regime, it is presumably the energy flux to the one third power, but knowledge of the exact physical nature of the flux is not necessary to test the prediction of cascade processes, eq. 2. First we decompose the wind into longitudinal and transverse
components anticipating that the altitude fluctuations will affect each somewhat differently. Fig. 2b shows the results along with those of the pressure. Although the basic structure is reasonably close to (near) planetary scale cascades, we can see evidence of the three regimes discussed above; at scales smaller than about 4 km, the variability is a little too large (compared with the regression lines), $\log M_q$ is quite linear from 4 – 40 km, and then flattens out a bit at scales > 40 km. Quantitatively this can be understood by the action of inertially smoothed trajectories at small but nonzero slopes (scales < ≈ 4 km), by turbulent wind induced altitude fluctuations for scales between 4 and 40 km and then at larger scales an excess of variability since the aircraft starts to follow the isobaric slopes and responds to the (larger) vertical intermittency (at 40 km the statistics are still quite good and we see a systematic departure above the regression lines, not just noise).

For the pressure, on the contrary the variability at the largest scales is low since the aircraft doesn’t stray far from the isobars. We may also note that for the wind, the outer scales are somewhat larger than the planetary scales (table 1), especially for the longitudinal component. In [Lovejoy et al., 2009c], we have already argued on the basis of cross-spectral analysis that the longitudinal component was the most affected by the trajectory; it is presumably partially responsible for the large value of $L_{eff}$ which imply an excessively large variability at all scales. In comparison, the value of $L_{eff}$ for the transverse component – although perhaps still too large - is about the same as that reported for satellite radar reflectivities from precipitation [Lovejoy et al., 2008a]; the fact that it is larger than the largest great circle distance (20,000 km) simply means that even at planetary scales there is residual variability due to the interactions of the wind with other atmospheric fields. In comparison the outer scale of the pressure is somewhat
smaller than planetary scales; this is not surprising since the aircraft was attempting to fly along isobars.

Fig. 2c: Same as fig. 2b but for the fields strongly affected by the trajectories: the longitudinal wind (top left $a$), the transverse wind (top right, $b$), pressure (c, lower left right).

Turning our attention to the analyses of the relatively unbiased temperature, humidity and log potential temperature fields (fig. 2d), we again see evidence for the three regimes and we note that the cascade structure is somewhat more closely followed with external scales somewhat smaller than 20,000 $km$. Overall, we conclude that the
basic predictions of the cascade theories are well respected; in particular there is no
evidence for a break anywhere near the meso-scale ($\approx 10 \text{ km}$, no sign of a 2D/3D transition). The outer scales of these fields are in fact very close to those of visible, infra
red and passive microwave radiances as determined by satellite [Lovejoy et al., 2009a]. Note that the log equivalent potential temperature is not shown here or below because the humidity is sufficiently low at the aircraft altitude that the graph is nearly indistinguishable from that of the log potential temperature.
Fig. 2d: Same as 2c but for the fields that are relatively unaffected by the trajectory: temperature (top left), relative humidity (top right), log potential temperature (lower left right).

### 3.2 Fluctuation analysis using structure functions:

In section 3.1 we discussed the statistics of the turbulent flux, we now turn our attention to the (absolute) fluctuations as functions of scale ($\Delta x$), here estimated using differences: e.g. $\Delta v(\Delta x) = |v(x + \Delta x) - v(x)|$. Defining the $q^{th}$ order structure function $(S_q)$:

$$S_q(\Delta x) = \left\langle (\Delta v(\Delta x))^q \right\rangle$$  \hspace{1cm} (6)a

we can now take the ensemble average of eq. 1 and obtain:

$$S_q(\Delta x) \propto \Delta x^{\xi(q)}; \quad \xi(q) = qH - K(q)$$  \hspace{1cm} (6)b

The difference between the structure function and the fluxes can be clearly be seen in the special case of 1-D analyses where fluxes at the smallest scale are estimated by absolute first order differences (as described in section 2). In this case, the large scale fluxes are obtained by adding/integrating the absolute small scale differences. In the case of structure functions, the large scale fluctuation is given by adding/integrating the signed small scale differences. In terms of scaling, the former yields an exponent $K(q)$ (with respect to $\lambda$) while the latter has exponent $Hq - K(q)$ (with respect to $\Delta x = L_{outer}/\lambda$). Since the linear term $Hq$ is frequently larger than the nonlinear term $K(q)$ (associated with intermittency) studying the fluxes is a more sensitive way of investigating the cascade, intermittency behaviour.
The results for altitude are given in fig. 3a. We again see evidence for three regimes; the large scale regime very nearly corresponds to linear variations ($H = 1$), i.e. to a constant slope with the aircraft nearly exactly following the isobars. As we shall see in more detail in the next section - where we estimate $H$ as a function of scale from the logarithmic slopes of fig. 3a - $H$ decreases at the smallest scale from a value $\approx 1$ corresponding to inertially smoothed trajectories, to a minimum value $\approx 0.5$ at around 4 km, and then increases to the maximum value 1 for scales $> 40$ km where the trajectories are close to isobars and are again, relatively smooth. Note that in fig. 3a, b, c we plot the nondimensional structure functions obtained by dividing the fluctuations by the mean at the smallest scale.
Fig. 3 a: Structure functions of order $q = 0.2, 0.4, ..2.0$ for the aircraft altitude. Distances $\Delta x$ are in $km$. Separate sets of regression lines are shown for both small and large scales in order to emphasize the change with scale.

In fig. 3b we show the corresponding results for the longitudinal and transverse wind and pressure. While the behaviour for the wind has essentially two scaling regimes with a $4 - 40$ $km$ transition regime – unsurprisingly - the pressure has poor scaling; we reserve a detailed analysis for the next section. In fig. 3c we show the corresponding plot for those fields less affected by the trajectory fluctuations, the temperature, humidity and log potential temperature; we see that the scaling is indeed very good.
Fig. 3 b: Same as fig. 3 a but for the fields strongly affected by the aircraft trajectory: the longitudinal wind (upper left), the transverse wind (upper right) and the pressure (lower left).
4. Refined analysis:

4.1 horizontal exponent estimates

4.1.1 The intermittency exponents and the optimum scale range

In section 3 we considered the raw flux statistics; no attempt was made to quantitatively correct or take into account the effects of the trajectory, we turn to this task.
now. From the “raw” flux and fluctuation analyses, we saw that the aircraft trajectories are highly intermittent for scales < 40 km (fig. 2a) after which they begin to accurately follow isobars and have nearly constant slopes (fig. 2b). We also saw that for the fields not strongly affected by the trajectory (temperature, humidity, log potential temperature), that the overall scaling was very good. However, if we want to quantitatively characterize the corresponding horizontal scaling and estimate $K(q)$ and hence the parameters $H, C_1, \alpha$, it is not clear what is the optimum range of scales that should be used. In order to obtain a more exact picture of the scale by scale variations in the statistics, we first consider scale by scale estimates of the basic flux and intermittency parameters using the logarithmic derivatives of the second order flux exponent ($K(2)$) and first order ($q = 1$) structure function exponent $H$. $K(2)$ was chosen to characterize the amplitude of the intermittency rather than the more fundamental $C_1$ because it is directly related to the spectral exponent $\beta = 1 + 2H - K(2)$ and so can be directly used to determine the contribution of intermittency corrections to the spectrum (note that since $\alpha \approx 2$ we have $C_1 \approx K(2)/2$, eq. 4; in any case, $C_1$ is directly estimated below using $C_1 = K'(1)$).
Fig. 4a: Second order trace moments (right column) and their logarithmic derivatives (estimates of $K(2)$, an intermittency index), left column. Top row: wind, longitudinal (blue) transverse (red). The region within the dashed lines has particularly low $K(2)$ ($\approx 0.05$). Middle row: pressure (blue), altitude (red). Bottom, temperature (blue), humidity (red), log potential temperature (gold), reference lines $K(2) = 0.12$. $\log_{10} \lambda = 0$ corresponds to 20,000 km ($= L_{ref}$).

Fig. 4a shows the results for the second moment of the flux $M_2$; the right column for the moments themselves, the left for the corresponding logarithmic derivatives estimated from the corresponding right hand graphs by performing regressions over an
octave of scales centred on the scale indicated. Focusing on the top row for the wind, we see that the logarithmic derivatives bring out the three regions quite clearly. At small scales $< \approx 4 \text{ km}$ (large $\lambda$), the intermittency is large corresponding to the strong intermittency of the trajectory while at the largest scales $\gg 40 \text{ km}$ (small $\lambda$) it is again large due to the fact that the near constant slope means that the aircraft is picking up the (large) vertical intermittency indeed, if we use the vertical value $K(2) \approx 0.17$ estimated from the drop sondes we find that it is not far from the value of the upper left (velocity) graph in fig. 4 a. Note that the estimate of the logarithmic slope is itself highly fluctuating at low $\lambda$’s since the statistics get progressively worse at large scales due to the smaller and smaller number of large scale structures. Detailed consideration of the scale by scale estimates of $K(2)$ for the altitude show that it is actually not so constant in spite of the relatively straight appearance of the lines in fig. 2 a. For the velocity, the region 4 - 40 km is the least intermittent (in the sense of the lowest exponent $K(2)$); this is presumably the optimum region for estimating the scaling exponents: at smaller scales, the aircraft trajectory is too intermittent whereas at the larger scales, the intermittency increases again since – at least for the wind - the aircraft is moving significantly in the vertical along a near constant isobaric slope (in any case at the large scales, we obtain the vertical rather than the horizontal exponents and the vertical intermittency is larger, see below and fig. 2 c). We therefore used this regime to estimate the exponents (table 2 estimated by taking regression over the range 4 – 40 km). We see that in conformity with the cross-spectral analyses and our previous discussion, that the scale by scale intermittency (i.e. $K(2) = d\log M_q/d\log \lambda$) for $T, \log \theta$ and $h$ is nearly constant for scales $\gg 4 \text{ km}$ confirming that the main deviation from scaling for these fields is only at the
smallest scales. We also observe that the exponents are nearly the same for all three fields. In table 2 we also give the corresponding $C_1$ estimates obtained by numerically estimating $K'(1)$ using regressions over the range 4 – 40 km.

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<th>$h$</th>
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<td>0.063</td>
<td>0.051</td>
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<td>0.042</td>
<td>0.028</td>
</tr>
<tr>
<td>$C_{1,mean}$</td>
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<td>0.052±0.010</td>
<td>0.040±0.012</td>
</tr>
<tr>
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<td>0.50</td>
<td>0.52</td>
</tr>
<tr>
<td>$H_{\beta}$</td>
<td>0.51</td>
<td>0.52</td>
<td>0.50</td>
</tr>
<tr>
<td>$H_{mean}$</td>
<td>0.50±0.01</td>
<td>0.51±0.01</td>
<td>0.51±0.01</td>
</tr>
</tbody>
</table>

Table 2: Parameter estimates over the “optimum” range 4 to 40 km. $\beta$ is the spectral exponent, $H_{st}$ is from the first order structure function, $H_{\beta} = (\beta + K(2)-1)/2$, and $H_{mean}$ is the average of the two. $C_1 = K'(1)$ from the trace moments and $C_{1,st} = \xi(1) - \xi'(1)$, $C_{1,mean}$ is the average.

4.1.2 The fluctuation exponent $H$:

Turning our attention to the fluctuation exponent $H$, (fig. 4 b) we see (top row, right, see the reference lines) that they reasonably follow successively the values $H = 1/3$, 0.75 (i.e. the standard Kolmogorov value and the observed vertical value respectively). However the scale by scale estimates (top, left) show that this mean behaviour hides an
increase in $H$ from $\approx 1/3$ at the smallest scales to $H \approx 0.75$ at scales $40 \text{ - } 100 \text{ km}$.

According to the leg by leg analyses in [Lovejoy et al., 2009c], this fairly systematic increase of ensemble mean $H$ hides highly variable transition scales in individual legs; the observed fairly continuous change in $H$ is in fact the result of the transition scale varying greatly from leg to leg. On the structure function (upper right) we can also see that fitting a single line through the entire range is not so bad, and yields roughly the mean of $1/3$ and $0.75$; this may explain the wind value $H \approx 0.56 \pm 0.02$ from [Hovde et al., 2009] (using nearly the same data set as here). This can be compared to the stratospheric estimates over (different) fractal isomach trajectories of $0.50 \pm 0.02$, $0.52 \pm 0.03$ (transverse, longitudinal wind respectively, [Lovejoy et al., 2004]) and $0.53 \pm 0.01$ [Tuck et al., 2004]. Note that the latter finds significant differences in $H$ for the wind for stratospheric trajectories across or along the polar jet ($0.45 \pm 0.14$ and $0.55 \pm 0.12$ respectively) but this could be a reflection of a systematic shift in the transition scale from an $H \approx 1/3$ to an $H \approx 0.75$ regime (see however the interpretation in [Tuck et al., 2004] and [Tuck, 2008], which is consistent with the thermal wind equation).

Moving down to the second row in fig. 4b, we see that at small enough scales, the altitude has $H_{tr} \approx 1$ presumably as a consequence of the inertial smoothing leading to near linear behaviour (nearly constant slopes; the subscript “$tr$” is for “trajectory”). At larger scales, the inertia effect is gradually overcome by turbulence so that at $4 \text{ km}$ it reaches a minimum near $H_{tr} \approx 0.48$ after which it systematically rises again to the smooth trajectory $H_{tr} \approx 1$ (for scales $\approx 40 \text{ km}$) as the aircraft flies increasingly close to a constant isobaric slope. Up to this scale, the altitude statistics are thus close to those of the ER-2 stratospheric trajectories except that the latter maintain a value $H_{tr} \approx 0.58$ along isomachs
(over the range ≈ 3 km until ≈ 300 km when the systematic vertical rise of ≈ 1 m/km (due
to fuel consumption) eventually implies $H_v \approx 1$ (on average, for scales ≳ 300 km;
[Lovejoy et al., 2004]). As noted above, along this fairly wide fractal range, the ER-2 had
an “anomalous” wind exponent $H_{\text{wind}} \approx 0.5$ which is nearly the mean of the horizontal and
vertical values (1/3, 0.75) so that the effect of following a fractal trajectory with fairly
well defined fractal dimension ($= 1+H_v$) apparently leads the ER2 to sample both
horizontal and vertical exponents yielding overall an average of the two.

Consider now the pressure exponent. At the smallest scales, it starts off with a
value near 2/3 decreasing systematically to zero at scales ≳ 40 km indicating very low
fluctuations consistent with near isobaric trajectories (there are some large fluctuations in
the scale by scale estimates presumably due to poor statistics). The small scale value ≈
2/3 is presumably a consequence of the dynamic pressure relation $\Delta p \approx \rho \Delta v^2/2$ so that
$H_{\text{press}} = 2H_{\text{wind}}$ with $H_{\text{wind}} \approx 1/3$.

Finally, on the bottom row of fig 4b, we show $H$ estimates for $T$, log$\theta$, $h$, again
finding relatively good scaling (constant exponents) for scales ≳ 4 km with $H \approx 0.50$ for
all three (see table 2). For temperature, this is also close to the value 0.52±0.02 obtained
from the similar tropospheric data [Hovde et al., 2009] and also not far from
stratospheric ER-2 analyses with $H_{\text{temp}} \approx 0.45\pm0.02$, [Lovejoy et al., 2004], and $H_{\text{temp}} \approx
0.54\pm0.01$ [Tuck et al., 2004]. This value is so close to a ratio of small integers (1/2) that
one would expect there to be a straightforward theoretical explanation (or at least
dimensional analysis) leading to this value but we are not aware of any adequate theory.
The usual argument is to consider temperature to be a passive scalar in which case
Corrsin-Obukhov theory predicts $H = 1/3$ (see Tuck 2008 for arguments about the nature
of atmospheric temperature that imply it is not a passive scalar). Alternatively, consider the prediction of Bolgiano-Obkühov theory for isotropic buoyancy driven turbulence, which for the wind predicts $H = 3/5$ which - recalling $\beta = 1 + 2H - K(2)$ with $K(2) \approx 0.05$ - is the vertical value observed near the surface ($\approx 2.15$, see section 1.1). However the same theory predicts $H_{\text{temp}} = 1/5$ which is even further from the empirical value (see [Monin and Yaglom, 1975]). Further support for the non standard value $H \approx \frac{1}{2}$ for temperature (corresponding to $\beta \approx 1.9$) comes from aircraft temperature spectra in the lower troposphere [Chigirinskaya et al., 1994] where $\beta \approx 1.9$ fits very well over the range $> 1$ km in scale, and stratospheric potential temperature spectra where $\beta \approx 1.9$ is also very close to data for scales $> \approx 3$ km in scale [Bacmeister et al., 1996] (recall that $K(2) \approx 0.1$ so that $\beta = 1 + 2x0.5 - 0.1 = 1.9$).
Fig. 4b: First order structure functions (right column) and their logarithmic derivatives (estimates of $H$), left column. Top row: wind, longitudinal (blue) transverse (red), the reference lines correspond to the exponents 1/3, 0.75. Middle row: pressure (blue), altitude (red), the reference lines correspond to $H = 1$, i.e. constant mean slopes. Bottom, temperature (blue), humidity (red), log potential temperature (gold), reference lines $H = 0.5$. 
4.1.3 The spectral exponent $\beta$:

Finally, we consider the spectra (fig. 4 c). As indicated above, since the spectrum is a second order statistic, we expect that the scale by scale analysis of the latter (again by logarithmic derivatives) should at least roughly satisfy the equation $\beta = 1 + 2H - K(2)$ (actually, this equation is strictly only valid when the scaling is satisfied over wide ranges so hence some deviations are to be expected; note that to estimate the spectrum, we used a standard Hanning window). Starting with the wind (the top row), we see the spectral version of the transition discussed earlier: at low wavenumbers ($k < (40 \text{ km})^{-1}$), $\beta \approx 2.4 (= 2 \times 0.75 - 0.1)$ to $\beta \approx 1.6 (= 2 \times 1/3 - 0.05$ for $(40 \text{ km})^{-1} > k > (4 \text{ km})^{-1}$). In the high wavenumber regime, we see a new feature which is a slight bump near $k \approx (1 \text{ km})^{-1}$, (magnified in the derivative on the left) which may be a residual signature of various aircraft roll modes and autopilot feedbacks. Turning our attention to the middle row (altitude, pressure), we see that the small scales are indeed particularly smooth with the larger scales following near Kolmogorov ($\beta \approx 5/3$) behaviours (probably the maximum near $(200 \text{ km})^{-1}$ is only a statistical fluctuation). Interestingly, the regime with reasonable $\beta \approx 5/3$ scaling seems to be a consequence of the partial cancellation of a continuously varying $H$ and $K(2)$ (compare the corresponding plots of 3a, 3b). Finally, the temperature, humidity and log potential temperature (bottom row) is seen to have significant fluctuations (left), but the overall scaling (far right) is nevertheless fairly good.

We now consider the parameter estimates (table 2), taken from the “optimum” regime 4 – 40 km. For both the $C_1$’s and $H$’s there are two slightly different ways to estimate them. $H$ can be estimated from both structure functions: $H_{st} = \xi(1)$ and also
from the spectral exponent $\beta$ via: $H_\beta = (\beta - 1 + K(2))/2$. The $C_1$ can be estimated from the flux moments: $C_1 = K'(1)$ and also from the structure function exponent: $C_1 = \xi(1) - \xi'(1)$.

In the table we have given all these estimates as well as the mean of the two as the best guess, and with the error as half the difference. From the table we see that the three fields have nearly identical exponents, and that the important parameter $H$ - which should ultimately be determinable by dimensional analysis - is near the value 1/2.

In fig. 5 we show the moment scaling exponents $K(q)$ estimated over the 4 - 40 km range. We see that $T$, log$\theta$, $h$ remain almost indistinguishable out to very large $q$ values (corresponding to rare extreme fluxes) whereas the longitudinal and transverse components separate for $q \approx 3$ (already a large value of $q$). The corresponding asymptotic linearity is predicted as a consequence of a “multifractal phase transition”, [Szépfalusy et al., 1987], [Schertzer et al., 1993] and depends on the finite size of the sample (the asymptotic slope is simply the largest singularity present in the sample).
Fig. 4 c: Spectra (right column) and their logarithmic derivatives (estimates of $-\beta$), left column. Top row: wind, longitudinal (blue) transverse (red). The thin reference lines correspond to $\beta = 5/3$, 2.4. Middle row: pressure (blue), altitude (red) with reference lines corresponding to $\beta = 5/3$. Bottom, temperature (blue), humidity (red), log potential temperature (gold), reference lines $\beta = 2$. Vertical dashed lines indicate the region with scales 4 – 40 km where the velocity intermittency is low.
4.2 Comparison with reanalyses and numerical models

In the introduction, we mentioned that the same basic horizontal cascade structure was found in reanalyses and in numerical models of the atmosphere (see the example fig. 1e and [Stolle et al., 2009]). It is therefore of interest to compare the cascade parameters with those found for the aircraft data studied here. Before doing so, it should be mentioned that the turbulent fluxes estimated for the spatial cascade analyses is based not on scaling range estimates (as here, see eq. 1) but rather on the model (hyper) dissipation scale estimates (using absolute numerical Laplacians). As pointed out in [Stolle et al.,
the fluxes estimated in the two different ways will in general be different, in the simplest case, assuming the standard turbulence results for the velocity field with $k^{-5/3}$ horizontal spectrum, the two fluxes are related by a power law (exponent $\eta = 3/2$) so that for universal multifractal cascades, the $C_1$ parameters are estimated by $C_{1\text{diss}} / C_{1\text{scaling}} = (\eta)^{\alpha}$ and with $\alpha \approx 1.8$, we find a ratio $\approx 2.08$ in this basic intermittency parameter (the parameter $\alpha$ should be the same). Although in general the relation between the fluxes and the parameters will be more complex, the ratio 2.08 turns out to be reasonable estimate).

In table 3, we compare the best aircraft and model estimates. Overall, we see that there is surprisingly close agreement.

<table>
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<th>$C_1$</th>
<th>$\alpha$</th>
<th>$L_{\text{eff}}$</th>
<th>$\delta$</th>
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<tr>
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<td>1.66±0.10</td>
<td>25000</td>
<td>0.6</td>
</tr>
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<td>(0.083), 0.040</td>
<td>1.81</td>
<td>10000</td>
<td>0.5</td>
</tr>
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</table>

Table 3: The optimum parameters at the 200 mb level (roughly the aircraft altitude, bold face) are given for the temperature, zonal wind and humidity for the analysis of the ERA40, GFS and GEM reanalysis, models (estimated from the model hyper dissipation scales up to 5000 km, taken from table 2 of [Stolle et al., 2009]). The principle value is the mean and the “±” indicates the spread about the mean. For $h$ (200 mb) the model values for $L_{\text{eff}}$ were 10,000 (GFS), 33,000 (GEM), 50,000 (ERA40) so that the geometric mean but no spread is given. For the $C_1$ aircraft estimates, the second value is from table 2, the first (in parentheses) is the same but increased by the factor 2.08 as a rough attempt to correct for the difference in the fluxes estimated in the scaling and dissipation regimes.
4.3 **Horizontal - Vertical comparison, vertical stratification, estimating $H_z$, $D_{el}$:**

We have argued that atmospheric fields are compatible with wide range horizontal scaling and have estimated the corresponding exponents. This work complements that of [Lovejoy et al., 2009b] where high resolution drop sondes were used to estimate the corresponding vertical cascades and exponents; the overall conclusion is thus that the dynamics are scaling and turbulent but anisotropic. The simplest anisotropic turbulence model involves a unique scale function for all the fields. This would imply that the ratio of horizontal and vertical components is $H_{hor}/H_{ver} = H_z = \text{constant}$, so that for universal multifractals $\alpha_{hor} = \alpha_{ver}$ and $C_{1hor}/C_{1ver} = H_z$. In [Lilley et al., 2008] there was an extensive analysis of this for the lidar backscatter data ($B$) from passive pollutants. Recall that the significance of $H_z$ is that it controls the scaling of the aspect ratio of structures in vertical sections. In particular, if we assume horizontal isotropy, then volumes of structures vary as $L^{2+H_z}$ where $L$ is their horizontal extent, hence the “elliptical dimension” $D_{el}$ controlling the rate of change of volumes of nonintermittent structures is $D_{el} = 2+H_z$. The 23/9 D model derives its name from $H_z = 5/9$, the result of Kolmogorov scaling ($H = 1/3$) in the horizontal and Bolgiano-Obukhov scaling in the vertical ($H = 3/5$). In comparison, the popular quasi-linear gravity wave theories ([Gardner, 1994], [Dewan, 1997]) have $H = 1/3$ in the horizontal and $H = 1$ in the vertical so that $H_z = 1/3$ and $D_{el} = 7/3$.

Combining the results from the aircraft and the drop sondes and taking into account a small apparent altitude dependence of the sonde intermittency exponents $C_1$ (so as
estimate them at the 200 mb aircraft level), we obtain table 3. Note that the value for the horizontal wind is given as the theoretical value 1/3 whereas the detailed leg by leg analysis in [Lovejoy et al., 2009c] which fits two power laws (one for small, the other for large scales) gives 0.26±0.07, 0.27±0.13 for the dominant small scale exponent (transverse and longitudinal respectively). Given the strong coupling between the aircraft trajectory and the wind, it seemed best to assume that these estimates support the Kolmogorov value 1/3 and it was used without an error bar. Similarly, the value of $H_z$ for velocity is from the analysis of [Lovejoy et al., 2009c] which fits two power laws one fixed at $H_h = 1/3$, the other (large scale one corresponding to $H_v$) varying.

It should be noted that although in table 3 we give the ratio of the $C_1$ values, since their values are small, their relative errors are large and consequently their ratios have large uncertainties. Since the $H$’s are larger, the ratio $H_z = H_h/H_v$ is more reliable than $C_{1h}/C_{1v}$, indeed in the latter case the error is very hard to reliably estimate and is not indicated except in the lidar case. The main conclusions are a) for $T$, log $\theta$ and $v$, the exponent ratio $H_z$ is close to 0.47, b) for $T$, log $\theta$, $B$, $H_z$ is within one standard deviation of the 23/9 D model value $H_z = (1/3)/(3/5) = 5/9$, c) the value for the humidity is somewhat larger. Note that finding virtually identical exponents for $T$ and log$\theta$ is not surprising since if the pressure is exactly constant, then the two have a one to one (albeit nonlinear) relation. Finally, also using the Gulfstream 4 data, [Hovde et al., 2009] find $H = 0.45\pm0.03$ for the humidity (see also Tuck 2008 for a possible explanation for the low value).
If the ratios in table 3 are taken at face value then we are lead to the conclusion that two or possibly three scale functions are required to specify the scale of atmospheric structures. While this is certainly possible, let us for the moment underline the various difficulties in obtaining the in situ estimates: the nontrivial vertical dropsonde outages, the nontrivial aircraft trajectory fluctuations. In addition, recall that detailed analysis of the altitude dependence of the horizontal velocity exponent in [Lovejoy et al., 2007] indicates that starting with the theoretical Boligano - Obukhov value 3/5 near the surface, that the exponent increases somewhat with altitude to the value $\approx 0.75$ at 10 - 12 km.

Similarly, the humidity (and hence log$\theta_e$) exponents may have both horizontal and vertical variations which may account for their high $H_z$ values (recall however that the values of $H$ are expected to be determined by dimensional analysis on fluxes; they are not expected to have truly continuous variations). We should therefore regard these studies as only early attempts to quantify the stratification.

<table>
<thead>
<tr>
<th></th>
<th>$T$</th>
<th>$\log \theta$</th>
<th>$h$</th>
<th>$v$</th>
<th>$B$</th>
</tr>
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<tbody>
<tr>
<td>$\alpha_h$</td>
<td>1.78</td>
<td>1.82</td>
<td>1.81</td>
<td>1.94</td>
<td>1.83</td>
</tr>
<tr>
<td>$\alpha_v$</td>
<td>1.70</td>
<td>1.90</td>
<td>1.85</td>
<td>1.90</td>
<td>1.82</td>
</tr>
<tr>
<td>$H_h$</td>
<td>0.50±0.01</td>
<td>0.51±0.01</td>
<td>0.51±0.01</td>
<td>1/3</td>
<td>0.33±0.02</td>
</tr>
<tr>
<td>$H_v$</td>
<td>1.07±0.18</td>
<td>1.07±0.18</td>
<td>0.78±0.07</td>
<td>0.75±0.05</td>
<td>0.60±0.02</td>
</tr>
<tr>
<td>$H_z = H_h/H_v$</td>
<td>0.47±0.09</td>
<td>0.47±0.09</td>
<td>0.65±0.06</td>
<td>0.46±0.05</td>
<td>0.55±0.02</td>
</tr>
<tr>
<td>$C_{1h}$</td>
<td>0.052±0.012</td>
<td>0.052±0.010</td>
<td>0.040±0.012</td>
<td>0.04</td>
<td>0.076</td>
</tr>
<tr>
<td>$C_{1v}$</td>
<td>0.072</td>
<td>0.071</td>
<td>0.091</td>
<td>0.088</td>
<td>0.11</td>
</tr>
<tr>
<td>$H_z = C_{1h}/C_{1v}$</td>
<td>0.72</td>
<td>0.71</td>
<td>0.44</td>
<td>0.45</td>
<td>0.69±0.2</td>
</tr>
</tbody>
</table>

Table 4: The above uses the estimate of the vertical $H_v$, $C_{1v}$ from sondes (table 2 in [Lovejoy et al., 2009b]). The (horizontal) values for $H_h$ for $T$, log$\theta$, $h$ are from table 2 (from 4 – 40 km, see above), for the lidar reflectivity $B$ it is from [Lilley et al., 2008]. Finally, the $C_1$ for $v$ is for the range 4- 40 km using trace moments. The horizontal $\alpha$ values were for nonlinear fits of the $K(q)$ for 0$<q<$3 (same range of scales).
5. Conclusions:

Satellites and other remote methods make the determination of the scale by scale statistical properties of atmospheric radiances relatively easy to establish. In comparison our knowledge of the corresponding variability of the standard atmospheric fields (wind, temperature humidity etc.) is quite meagre and is based almost entirely on problematic in situ measurements. The fact that today’s in situ atmospheric data is often orders of magnitude better than those available even twenty years ago - and is often easily accessible - has unfortunately encouraged naïve applications. For example, in a series of papers, we have argued that the extreme (and highly clustered) nature of data outages in drop sonde profiles must be carefully taken into consideration when they are analysed \cite{Lovejoy2009b}. In situ aircraft data provide another relevant example: not only do aircraft now have about 10 times higher resolution than they did in the 1980’s (e.g. 3 km for GASP, 280 m for the Gulfstream 4), more importantly, their altitude data is sufficiently accurate so as to permit a quantitative study of the relation between the aircraft altitude, pressure and turbulence. In a companion paper, we used this information to argue that due to strong coupling over wide ranges of scale between the wind and the trajectories, that aircraft measurements require a theory of anisotropic scaling turbulence for their interpretation \cite{Lovejoy2009c}. Whereas the classical interpretation of aircraft wind statistics assumes that the turbulence is isotropic and interprets the observed break at around 40 km as a transition between two isotropic regimes, in the reinterpretation, it is spurious and is simply a transition from horizontally to vertically dominated parts of the trajectory.
In this paper, we have attempted to exploit this new understanding in order to both demonstrate and to quantify the wide range horizontal scaling of the key atmospheric fields: wind, temperature, humidity, potential temperature. The basic theoretical framework for such wide range scaling is anisotropic cascades; we demonstrate the horizontal cascade structure by directly analyzing the raw turbulent fluxes finding that they all follow a multiplicative cascade structure with external scales of the order of 10,000 km. Although this basic structure was present in all the analyzed fields, we noted the presence of the three different regimes predicted on the basis of their cross-spectra with the aircraft altitude and with the pressure. The deviations were fairly strong for the wind and pressure which were strongly coupled with the trajectory, but relatively small for the temperature, humidity and potential temperature which were only weakly coupled. A rough summary is that at scales $\approx 4$ km the trajectories vary smoothly but are affected by aircraft roll and pilot/autopilot controls, for scales $\approx 4 - 40$ km the aircraft follow a rough, intermittent fractal trajectory; finally for scales $\geq 40$ km the aircraft very closely follow the isobars which have significant vertical slopes. These regimes are for the ensemble statistics; the transition points vary considerably from trajectory to trajectory.

Refined analyses of the statistics was performed by considering the scale by scale intermittency (characterized by the second order moment of the flux) as well as the scale by scale smoothness of the fluctuations (characterized by the first order structure function). It was argued that the optimum scale range for parameter estimation was $\approx 4 - 40$ km where the intermittency in the altitude was not strong, the effect of the pilot/autopilot, rolls was weak and the constraint of flying on an isobar was not yet so
strong as to impose the verticals statistics on the wind field. We therefore used this range
to give refined parameter estimates. Perhaps one of the most surprising results was the
near identity of the scaling parameters $H$, $C_1$, $\alpha$ for temperature, log potential temperature
and humidity, and in particular the non standard result $H = 0.51 \pm 0.01$ for all three fields
which is extremely close to the fraction $\frac{1}{2}$ but which nevertheless (apparently) still belies
theoretical explanation (see however Tuck 2008).

Finally, we compared the estimates of the horizontal exponents with those from
drop sondes of the corresponding vertical exponents. Overall, the ratios $H_z$ for $v$, $T$, log$\theta$,
h were not far from the 23/9 D model prediction 5/9 (based on energy flux domination of
horizontal statistics and buoyancy variance flux domination of vertical statistics), yet
there were apparently systematic deviations: $v$, $T$, log$\theta$ had $H_z \approx 0.47$ and for $h$, $H_z \approx 0.65$.
For the moment there is no satisfactory theoretical explanation for these results, although
some clarification may come from a comparison with the structure of atmospheric
reanalyses and (hydrostatic) numerical models which not only have nearly perfect
cascade structures in the horizontal, but also at least some similar (and non-standard)
parameters including - at least for the ECMWF interim reanalysis – nearly identical
spatial exponents for the isobaric wind exponents and the vertical sonde exponents. The
fact that we find qualitatively (and in many cases quantitatively) similar wide range
cascade structures in the simulations and in the atmosphere opens up new avenues for
statistically verifying the models: their scale by scale statistics should be the
quantitatively the same as those of the measurements.

It is remarkable that in spite of the current golden age of meteorological
observations that there is still no scientific consensus about the atmosphere’s basic scale
by scale statistical properties, in particular those of the dynamical (wind) field. However, if we reinterpret the (systematically observed) transition from $\approx 5/3$ to $\approx 2.4$ spectral scaling in terms of a single anisotropic scaling turbulence then there are no longer seriously obstacles to the emergence of a consensus. To complete this emerging “new synthesis” [Lovejoy and Schertzer, 2009] the analyses presented here must be extended to the time domain, to different altitudes, should be rechecked with different sensors and in particular to aircraft flying on isoheights. Perhaps most importantly, completion of the synthesis requires that we abandon the dogma of isotropy (or quasi isotropy) and embark upon the systematic development of anisotropic but scaling theories of turbulence.

6. Acknowledgements

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7. References


Stolle (2009), Space-time Cascade structure of numerical models of the atmosphere, McGill, Montreal.


