ESTIMATION OF UNIVERSAL MULTIFRACTAL INDICES FOR ATMOSPHERIC TURBULENT VELOCITY FIELDS

F. SCHMITT and D. SCHERTZER
Laboratoire de Météorologie Dynamique, Boîte 99,
Université Paris 6, 4 place Jussieu,
75252 Paris cedex 05, France

S. LOVEJOY
Physics Dept., McGill University, 3600 University St.,
Montréal, Qué., H3A 2T8, Canada

Y. BRUNET
INRA Bioclimatologie, 71 av. Edouard-Bourlaux, BP 81,
33883 Villeneuve d’Ornon cedex, France

Abstract
We study wind turbulence with the help of universal multifractals, using atmospheric high resolution time series. We empirically determine the three universal indices \( (H, C_1, \alpha) \) which are sufficient to characterize the statistics of turbulence. The first, \( H \), which characterizes the conservation of the field, is theoretically and empirically known to be \( \approx 1/3 \), while \( C_1 \) corresponds to the inhomogeneity of the mean field \( (C_1 = 0 \text{ for homogeneous fields, and } C_1 > 0 \text{ for inhomogeneous and intermittent fields}) \). The most important index is the Lévy index \( \alpha \) corresponding to the degree of multifractality \( (0 \leq \alpha \leq 2, \alpha = 0 \text{ for a mono fractal}) \). The two latter indices are directly obtained by applying the double trace moment technique \( (DTM) \) on the turbulent field. Analyzing various atmospheric velocity measurements we obtain: \( \alpha \approx 1.45 \pm 0.1 \) and \( C_1 \approx 0.25 \pm 0.1 \). These results show that atmospheric turbulence has nearly the same multifractal behavior everywhere in the boundary layer, corresponding to unconditionally hard multifractal \( (\alpha \geq 1) \) processes. This describes the entire hierarchy of singularities of the Navier-Stokes equations.
1. INTRODUCTION

In the limit of high Reynolds number, turbulence is known to be scale invariant.\textsuperscript{1,2} One of the points which has been important in characterizing fully developed turbulence is to know whether it belongs to universal classes which could describe any high Reynolds number turbulent flow.\textsuperscript{3,4} We test here such a scale invariant model of turbulence, called the universal multifractal model\textsuperscript{5,6} using different atmospheric turbulent datasets. This model is determined by only three indices, which completely describe the scaling behavior of the turbulent fields. We compare the values of these indices for various atmospheric datasets, in order to confirm the hypothesis of their constancy.

2. THE DATA AND THEIR SPECTRA

The datasets are turbulent velocity measurements made in the atmospheric boundary layer: the most important one was taken near Montreal, at 3 meters from the ground, with a high-resolution hot wire anemometer, and the others with sonic anemometers in Paris, and in Bordeaux (see Table 1 for a description of the experimental conditions).

<table>
<thead>
<tr>
<th>Dataset #</th>
<th>Anemometer</th>
<th>Sampling Frequency (Hz)</th>
<th>Place of Acquisition</th>
<th>Altitude (m)</th>
<th>Total Number of Data Points used</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Hot wire</td>
<td>2000</td>
<td>Montreal</td>
<td>3</td>
<td>720,000</td>
<td>1.70</td>
</tr>
<tr>
<td>2</td>
<td>Sonic</td>
<td>200</td>
<td>Paris</td>
<td>20</td>
<td>25,000</td>
<td>1.72</td>
</tr>
<tr>
<td>3</td>
<td>Sonic</td>
<td>10</td>
<td>Bordeaux</td>
<td>25</td>
<td>360,000</td>
<td>1.70</td>
</tr>
</tbody>
</table>

Their wind energy spectrum follows a power law over a wide range of scales:

\[
E_\nu(k) \approx k^{-\beta}
\]

with the rate of acquisition, the range of scaling, and \( \beta \) given in Table 1. Figure 1 shows the power spectrum of the dataset #1. The values of the spectral slopes are very close to the Kolmogorov-Obukhov phenomenological derivations predicting \( \beta = 5/3 \).\textsuperscript{1,2} We then assume the validity of the refined\textsuperscript{3} similarity hypothesis:

\[
\Delta \nu(\ell) \approx \varepsilon_\ell^{1/3} \ell^H
\]

where \( \Delta \nu(\ell) \) is the wind shear \(|\nu(x + \ell) - \nu(x)|\) at scale \( \ell \), \( \varepsilon_\ell \) is the local rate of energy transfer (from scales larger to scales smaller than \( \ell \)). The index \( H \) is the mean deviation from conservation of the velocity shears, and is related to \( \beta \) by \( \beta \approx 2H + 1 \) (e.g. see Ref. 7).

\textsuperscript{3}Kolmogorov 1941 assumed \( \varepsilon \) to be homogeneous in space, i.e., \( \varepsilon = \varepsilon_\ell \), whereas Eq. (2) takes into account the small scale intermittency of turbulence with the introduction of a very intermittent energy transfer field.
Fig. 1 The wind energy spectrum of atmospheric turbulence corresponding to dataset #1. For a wide range of scales, it follows a power law relation $E_\omega(k) \approx K^{-\beta}$ with $\beta \approx 1.68 \pm 0.05$ as predicted by Refs. 1 and 2.

Fig. 2 A pattern of $\varepsilon$, the rate of energy transfer from large to small scales, for atmospheric turbulence: this is a very intermittent (and scaling) field.

This gives us $H = 0.33 \pm 0.03$ for the three datasets, as predicted by phenomenological models. In order to study the statistical properties of the very intermittent dissipation field $\varepsilon_t$, we perform a fractional differentiation of order $1/3$ of the wind field (i.e., a multiplication by $k^{1/3}$ in Fourier space) and then take the third power: the result is a very intermittent field, whose pattern can be seen in Fig. 2 for atmospheric measurements corresponding to dataset #1.
3. UNIVERSAL MULTIPLE SCALING OF THE FIELD $\epsilon$

We have studied the energy flux $\epsilon$ using multifractal measures resulting from a multiplicative cascade process.\textsuperscript{4,8-11} When the cascade has proceeded over a scale ratio $\lambda = \frac{\ell}{\ell'}$ (the ratio of the largest scale of interest to the smallest scale) the density of the conserved energy flux has the singular behavior\textsuperscript{8,12,13}:

$$\epsilon_\lambda \approx \lambda^\gamma$$ \hspace{1cm} (3)

When $\lambda \to \infty$ (or $\ell \to 0$), $\gamma > 0$ is an order of singularity. The statistical moments of this field will have the following scaling behavior:\textsuperscript{5}

$$\langle (\epsilon_\lambda)^q \rangle \approx \lambda^{K(q)}$$ \hspace{1cm} (4)

where $\langle \rangle$ indicates ensemble averaging, and $K(q)$ is a nonlinear function\textsuperscript{b} which characterizes all the statistics of the field.

For universality classes (the stable and attractive limit obtained when mixing cascade processes, see Refs. 5 and 6), the functions $K(q)$ depend only on two indices:

$$K(q) = \begin{cases} \frac{C_1}{\alpha - 1} (q^\alpha - q) & \alpha \neq 1 \\ C_1 q \ln(q) & \alpha = 1 \end{cases}$$ \hspace{1cm} (5)

where the Lévy index $\alpha$ ($0 \leq \alpha \leq 2$) can be understood as an interpolation between the two extremes and well known cascade models of turbulence: the $\beta$-model ($\alpha = 0$)\textsuperscript{4,8,9} and the lognormal model ($\alpha = 2$)\textsuperscript{3,14,15}; and the second index corresponds to the mean inhomogeneity of the field: $C_1 = 0$ for homogeneous fields, and the larger $C_1$, the more the intermittent field. We have estimated these indices for wind tunnel and atmospheric energy fluxes, using a generalization of the (simple) scaling exponent $K(q)$.

4. ESTIMATION OF $\alpha$ AND $C_1$ WITH THE HELP OF THE DOUBLE TRACE MOMENT (DTM) ANALYSIS TECHNIQUE

The basic idea of the DTM technique\textsuperscript{16,17} is to generalize the application of statistical methods to the quantity $(\epsilon_\lambda)^\eta$. This is done by taking the $\eta$th power of $\epsilon_\lambda$ at the scale ratio $\lambda$ (the outer or largest scale of interest to the smallest scale of homogeneity), and then studying its scaling behavior at decreasing values of the scale ratio $\lambda \leq \Lambda$:

$$\epsilon_\lambda^{(\eta)} = \frac{(\epsilon_\lambda)^\eta}{\langle (\epsilon_\lambda)^\eta \rangle}$$ \hspace{1cm} (6)

The moments of this new field then have the following multiple scaling behavior:

$$\langle [(\epsilon_\lambda^{(\eta)})^q] \rangle \approx \lambda^{K(q, \eta)}$$ \hspace{1cm} (7)

where $K(q, \eta)$ is the $q, \eta$ double trace moment scaling exponent related to $K(q, 1) \equiv K(q)$ by $\langle [(\epsilon_\lambda)^q] \rangle$ is a constant:

$$K(q, \eta) = K(q \eta) - qK(\eta)$$ \hspace{1cm} (8)

\textsuperscript{b} $K(q)$ is related to the strange attractor notation\textsuperscript{13} by: $K(q) = (q - 1)D - \tau(q)$ where $D$ is the dimension of the space.
It gives for universality classes:

\[ K(q, \eta) = n^\alpha K(q) \quad (9) \]

By keeping \( q \) fixed (but different from the special values 0 or 1), the slope of \( |K(q, \eta)| \) as a function of \( \eta \) on a log-log graph gives the value of the index \( \alpha \), which with the help of the intersection with the line \( (\eta = 1) \) yields \( C_1 \). Varying \( \eta \) then allows for a systematic verification of Eq. (9), and hence the universality hypothesis.

Figure 3 shows \( \langle (\varepsilon_{\lambda, \lambda}^{(n)})^q \rangle \) vs. \( \lambda \) in a log-log plot for atmospheric energy flux corresponding to dataset #1: the straight lines show that the scaling of Eq. (7) is well respected, and their slope are the estimates of \( |K(q, \eta)| \). Figure 4 shows the curves log\( [K(q, \eta)] \) vs. log\( (\eta) \) for different values of the parameter \( q \), for atmospheric energy flux corresponding to dataset #1: the straight lines show that Eq. (9) is well respected for a wide range of \( \eta \)-values, and their slopes give the estimates shown in Table 2.

### Table 2

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Maximum Scale Ratio</th>
<th>( \alpha )</th>
<th>( C_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2048</td>
<td>1.50 ± 0.05</td>
<td>0.25 ± 0.05</td>
</tr>
<tr>
<td>2</td>
<td>1024</td>
<td>1.45 ± 0.1</td>
<td>0.29 ± 0.1</td>
</tr>
<tr>
<td>3</td>
<td>512</td>
<td>1.4 ± 0.1</td>
<td>0.24 ± 0.1</td>
</tr>
</tbody>
</table>
According to error estimates presented in Table 2, we propose the following values for the indices $\alpha$ and $C_1$ in the atmospheric boundary layer: $\alpha \approx 1.45 \pm 0.1$ and $C_1 \approx 0.25 \pm 0.1$. We note that these values are in the same range that estimates obtained for wind tunnel turbulence$^{18-20}$: $\alpha \approx 1.3 \pm 0.1$ and $C_1 \approx 0.25 \pm 0.05$. These values of $\alpha$ and $C_1$ show that turbulence is a hard multifractal process$^{18,21-23}$: because $\alpha > 1$, divergence of moments$^{4,10,20,24}$ is expected to occur.

5. MOMENTS OF THE ENERGY FLUXES AND STRUCTURE FUNCTIONS OF THE WIND FIELD

With the help of these results, we compare empirical statistical moments with $K(q)$ given by Eq. (5), in Fig. 5: for moments order $q < 3.0 \pm 0.5$ the empirical and “theoretical” scaling exponents are in excellent agreement. This critical moment$^{21,24}$ corresponds either to a second order phase transition analogue (a maximum moment computable due to the finite size of the dataset), or to a first order phase transition$^{27}$ analogue.

Equations (2) and (4) give the scaling velocity structure functions: $\langle [\delta u(i)]^{H} \rangle \approx \lambda^{-\zeta(q)} \approx \langle (\sigma)_{H}^{(n)} \lambda^{-H_{q}}$, which gives the relation $\zeta(q) = Hq - K(q)$. Thus, in the case of database #1, it is useful to study the structure function $\zeta(q)$ only up to moment order $3q_{max} \approx 9.0 \pm 1.5$, unlike Ref. 25, which computed structure functions up to order 18, and Ref. 26 which used the latter structure function to evaluate $\alpha$.  

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6. CONCLUSION

Using various atmospheric datasets, we obtain similar values of the three indices \((H, C_1, \alpha)\) which completely characterize the statistics of turbulence: \(H = 0.33 \pm 0.03\), \(C_1 \approx 0.25 \pm 0.1\) and \(\alpha \approx 1.45 \pm 0.1\) for the three atmospheric datasets studied. These results show that turbulence is a "hard" multifractal process \((\alpha \geq 1)\), and that it belongs to theoretically predicted universality classes.

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\(^d\)This divergence of moments is expected to occur because of some very violent "head" small scale singularities, which are not smoothed by integration: if the number of samples increases, more and more of such singularities are encountered, and the statistical moments diverges.
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