Prepared for the Proceedings of the Royal Society of Meteorology

SCALING TURBULENT ATMOSPHERIC STRATIFICATION, PART II: EMPIRICAL STUDY OF THE THE STRATIFICATION OF THE INTERMITTENCY

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ABSTRACT

We critically re-examine existing empirical vertical and horizontal statistics of the horizontal wind and find that the balance of evidence is in favour of the Kolmogorov $k_x^{-5/3}$ scaling in the horizontal, Bolgiano-Obukov scaling $k_z^{-11/5}$ in the vertical corresponding to a $D_5=23/9D$ stratified atmosphere. This interpretation is particularly compelling once one recognizes that the 23/9 D turbulence can lead to long range biases in aircraft trajectories and hence wind, temperature and other statistics. Indeed, we show quantitatively that one is able to reinterpret the major aircraft-based campaigns (GASP, MOZAIC) in terms of the model. In part I we have seen that this model is compatible with turbulence waves which can be close to classical linear gravity waves in spite of their quite different mechanism. We then use state-of-the-art lidar data of atmospheric
aerosols (considered a passive tracer) in order to obtain direct estimates of the effective
dimension of the atmosphere, the elliptical dimension, $D_{el}=23/9=2.55\pm0.02$. This result
essentially rules out the standard 3D or 2D isotropic theories which have $D=3,2$
respectively. In this paper we focus on the multifractal (intermittency) statistics showing
that to within experimental uncertainty, the high and low order statistics are indeed
stratified in the same way as the statistics near the mean field. We also show that they
are well approximated by universal multifractals; we estimate the exponents to be $\alpha_h$
$=1.82\pm0.05$, $\alpha_v =1.83\pm0.04$ $C_{1h}=0.037\pm0.0061$ and $C_1=0.059\pm0.007$ (horizontal and
vertical respectively). Since to within experimental accuracy, $\alpha_h = \alpha_v$ and
$C_{1h}/C_{1v}=H_2=5/9$ we conclude that the strong and weak regions (i.e. including the
intermittent structures) have the same scaling stratification as the mean field.

We examine the implications of the lack of scale separation in either the
horizontal or vertical directions implies for the Boussinesq and related approximations.

1. INTRODUCTION

In part I, we argued that the 23/9D model with the extension to unlocalized
propagators for the observable (e.g. velocity, density) fields provided the most physically
satisfactory model of the stratified atmosphere, being based on two turbulent fluxes, (the
energy and buoyancy force variance fluxes), respecting generalized Kolmogorov and
having some wave phenomenology. In this paper, we examine the corresponding
empirical evidence. In particular, we directly determine $H_2$ using 9 airborne lidar vertical
cross-sections of atmospheric aerosol covering the range 3m to 4500m in the vertical (a
factor of 1,500), and 100m to 120km in the horizontal (a factor of 1,200). One important difference between such airborne lidar measurements and in-situ aircraft measurements is that the former do not suffer from aircraft trajectory biases. This is because lidar is a remote sensing technique in which ground is used as the reference “altitude”. The key result of this experiment - announced in [Lilley, et al., 2004] - is convincing evidence for the 23/9D model. It yields $H_z = 0.55 \pm 0.02$ and therefore $D_{el}=2.55\pm0.02$ so that the 2-D and 3-D theories are well outside the one standard deviation error bars. These error bars are particularly small since each of the nine 2D sections have several orders of magnitude more data than the largest comparable balloon experiments (see table 1). Here, the aerosols act as a tracer, and laser light is scattered back to a telescope in the aircraft enabling a two-dimensional reconstruction of its spatial distribution. This in turns allows the determination of the degree of stratification of structures as functions of their horizontal extents. The horizontal range is particularly significant since it spans that critical 10km scale where the 3-D to 2-D transition – the mesoscale gap – has been postulated to occur. In addition, each data set is obtained within a short period of time (about 20 minutes) so that the meteorology is roughly constant. The result $D = 2.55\pm0.02$ is almost exactly that predicted from the 23/9 dimensional model and shows that even at scales as small as 3m the atmosphere does not appear to be three dimensional, nor at large scales does it ever appear to be perfectly flat (i.e., two dimensional). Rather, structures simply become more and more (relatively) flat as they get larger.

2. BRIEF REVIEW OF THE EMPIRICAL EVIDENCE:

2.1 Scaling in the vertical direction:

Although there is still no consensus about the nature of the empirical horizontal
spectrum (the 2D versus 3D debate for instance), in the vertical, there are fewer difficulties if only because it is easier for a single experiment to cover much of the dynamical range. The 23/9 D theory was motivated by the conclusions of the empirical campaign in Landes described in Section part I and by the radiosonde observations of horizontal wind shear along the vertical made by [Endlich, et al., 1969] and Jimsphere observations by [Adelfang, 1971]. At about the same time, [Van Zandt, 1982] proposed the anisotropic $k^{-5/3}$ (horizontal), $k^{-3}$ (vertical, gravity wave) theory, discussed in part I. Table 2 summarizes and compares some of the vertical studies (mostly of the horizontal wind or temperature). The most important general conclusion is the finding that $\beta_v > \beta_h$, i.e. there is no evidence of isotropic turbulence at any scale. Recall that $\beta_v > \beta_h$ implies a differential stratification becoming increasingly pronounced (flatter) at larger and larger scales. Although the interpretations of the campaigns were made from the perspective of various gravity wave theories, the actual spectral slopes (as Table 1 indicates) are in fact often closer to the Bolgiano-Obhukov value of 11/5 than the standard gravity wave value of 3.

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<td>An exponent ( \beta_v = 11/5 ) is compatible with the spectrum. One hour averaging. Low pass filtering. 3 km smoothing. An exponent ( \beta_v = 11/5 ) is compatible with the spectrum. One hour averages are made. 3 km smoothing applied. Low pass filter applied. Effort to isolate individual gravity wave events (quasi-monochromatic waves should not be confused with continuous spectra)</td>
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Table 1: A review of some of the existing empirical evidence for the scaling of atmospheric horizontal wind shear, temperature and density in the vertical direction ($\beta_z$). Note that these spectra were typically obtained for a very limited range of scales, often less than an order of magnitude. Our estimates of $\beta_z$ (far right column) we from the graphs in the published papers and are therefore somewhat subjective. When there is a large discrepancy between our estimates and the author’s estimates, it is due to a differing range of scales used in the estimate.

Note that the theoretical prediction of [Bolgiano, 1959] [Obukhov, 1959] is rarely discussed in comparison with other theories or with experimental results in the recent atmospheric experimental literature, possibly because of the belief that it is not compatible with wave phenomenology. Discussions related to the isotropic Bolgiano-Obukhov scaling on the effect of buoyancy, stratification and convection on the spectrum and the Bolgiano length $l_b$ at which the transition from 3D isotropic $k^{-5/3}$ turbulence to anisotropic 3D $k^{-11/5}$ turbulence can be found mostly in the buoyancy-driven Rayleigh-Benard laboratory experiments literature (see the discussion in [Lilley, et al., 2004]).

2.2 SCALING IN THE HORIZONTAL DIRECTION:

The early claims about the horizontal spectra (in particular the influential [Van der Hoven, 1957] spectra) were taken in the time domain and converted into horizontal
spatial spectra by using Taylor’s hypothesis of “frozen turbulence”. This assumption ([Taylor, 1938]) was originally made as a basis for analyzing laboratory turbulence flows in which a strong scale separation exists between the forcing and the turbulence; one simply converts from time to space using a constant (e.g. mean large scale) velocity assuming that the turbulent fluctuations are on such shorter time scales that they are essentially “frozen” with respect to the rapid advection of structures transported by the mean flow. In the atmosphere, the validity of this assumption depends on the existence of a clear large scale/small scale separation. The difficulties in interpretation are illustrated by the debate prompted by the early studies - especially [Van der Hoven, 1957] - which were strongly criticized by Goldstein (1968), [Pinus, 1968] and [Vinnichenko, 1969] and indirectly by [Hwang, 1970]. For instance, after commenting that if the meso-scale gap (separating the small scale 3D regime from the large scale 2D regime) really existed, it would only be for less than 5% of the time, [Vinnichenko, 1969] even noted that Van der Hoven’s spectrum was actually the superposition of four spectra – including a high frequency one taken under “near hurricane” conditions.

In order to obtain direct estimates of horizontal wind spectra, [Brown and Robinson, 1979] used the standard meteorological measuring network, but the scales were very large and intermittency was so strong that they could not obtain unambiguous results. A more direct way to obtain true horizontal spectra is to use aircraft data, and indeed, since the 1980’s, there have been two ambitious experiments (GASP, MOZAIC) to collect large amounts of horizontal wind data, both using commercial airliners fit with anemometers. The basic problem here is that aircraft are affected by turbulent updrafts and tail winds so that the interpretation of in-situ measurements themselves requires a
theory of turbulence. For example, if one accepts that the large scale is flat (2-D), then the vertical variability is small so that this predicts that deviations of the aircraft from a perfect straight-line horizontal trajectory will be small and that the effect of the turbulent motion on the aircraft is expected to be negligible. Similarly, if one is in an isotropic 3D regime, then there is only one exponent (the same in every direction) so that if one finds scaling, the exponent is naturally interpreted in terms of the unique scaling exponent of the regime.

In a recent paper [Lovejoy, et al., 2004], it was shown that due to the effects of anisotropic (presumably 23/9 dimensional) turbulence, aircraft can fly over distances of hundreds of kilometers on trajectories whose fractal dimension is close to 14/9 rather than 1, i.e. that are strongly biased by the turbulence that they measure. In this case, the long range bias was the result of using a “Mach cruise” autopilot that enforced correlations between the temperature and the aircraft speed such that the Mach number was nearly constant. In addition, it was pointed out that if in a 23/9 D turbulence, the aircraft does not fly at a perfectly flat trajectory but rather at an average slope $s$, then there will be a critical distance $\Delta x_c$ such that for larger distances, the statistics will be those of the vertical rather than the horizontal. Fig. 1a, 1b show that the wind statistics from GASP and MOZAIC - which are the two largest scale experimental campaigns to date - can readily be explained in the context of the 23/9 D model with only very small average aircraft slopes.
Fig. 3a: From the graph (adapted from [Lindborg and Cho, 2001]) we see that the aircraft inertial scale is $\Delta x_i = 20$ km, while the “linear rise scale” $\Delta x_f = 75$ km. If the tropospheric spherocape = 4 cm, then we find that an average slope of 1.5 m/km explains the entire MOZAIC structure function. The transition zone (possibly fractal) is roughly between 20 km, 75 km.
Fig. 3b: Reproduced from [Gage and Nastrom, 1986]. Red is a reference line with slope \(-5/3\), blue has slope \(-11/5\), yellow, \(-3\). Concentrating on the more reliable solid black lines which is the result from the data intensive GASP experiment (and ignoring the selected “turbulent episode” subset) we see that the BO blue line does an excellent fit from 20km on up. Once again, if the tropospheric spheric scale \(= 4\) cm, then we find that an average aircraft slope of roughly 1.5m/km explains the GASP spectra.

Not only does it seem that the 23/9 D theory is the only one that can account for these major horizontal spectral studies but results of satellite studies of cloud radiances provide additional support. Although cloud radiances are not directly related to the horizontal wind, the two fields are nonetheless strongly nonlinearly coupled such that if the scale invariant symmetry is broken in one, it will almost certainly be broken in the other. This was the motivation of the area-perimeter study mentioned above. More recently [Lovejoy, et al., 1993], [Lovejoy, 2001], [Lovejoy and Schertzer, 2005] have shown the existence of scaling in cloud studies extending to nearly 1000 satellite pictures in both
visible and infrared wavelengths. The latter directly showed that the radiance statistics are – to within 0.8% over the range of planetary scales down to 2km – were the same as those generated by a scale invariant cascade starting at planetary scale. It is not obvious how several different horizontal regimes could be hiding in this data. At the same time, multifractal cloud simulations (including those based on the turbulence/wave model) show how the strong horizontal anisotropy predicted by GSI (off diagonal elements in the horizontal part of $G$), can reconcile the wide diversity of cloud morphology, texture and type with the isotropic statistics which essentially wash out most of the anisotropy.

2.3 Lidar and Direct measurements of differential stratification:

During the 1980’s and 90’s there was growing evidence in favour of the 23/9 D model, this evidence was mostly indirect since vertical and horizontal statistics have almost invariably been studied in separate experiments in separate regions of the world and at different times. Until the lidar study [Lilley, et al., 2004], the only exceptions were the radar rain study [Lovejoy, et al., 1987] which only had a factor of 8 in scale in the vertical, and the roughly simultaneous aircraft radiosonde studies reported in Chiginiskaya et al 1994, Lazarev et al 1994. Direct tests of the fundamental prediction of differential stratification of structures have been lacking since they could only be obtained remotely by near instantaneous vertical cross-sections. Thanks to developments in high powered lidar – primarily the ability to digitize each pulse in real time with a wide dynamic range using logarithmic amplifiers - this type of data is now available. The lidar measures the backscatter ratio ($B$; the ratio of aerosol backscatter to background molecular scattering) of aerosols far from individual point sources; the measured
backscatter ratio is taken as a surrogate for the concentration of a passively advected tracer.

[Lilley, et al., 2004] used data which were taken as part of the PACIFIC 2001 airborne lidar experiment using an airborne lidar platform called AERIAL (AERosol Imaging Airborne Lidar) flown at a constant altitude over a grid of flight legs of up to 100km in the Lower Fraser Valley (British Columbia, Canada). Although the airborne lidar platform is a simultaneous up-down system mounted aboard the NRC-CNRC Convair 580 aircraft only data from the downward pointing system was used. The lasers operated at the fundamental wavelength of 1064 nm (suited for the detection of particles of the order of 1\(\mu\)m), with a pulse repetition rate of 20Hz. The output power of the downward lidar was measured to be 450mJ. The beam divergence was 6.6 mrad. The detectors employed were 35.6cm Schmidt-Cassegrain telescopes with an 8mrad field-of-view which focused the captured photons onto 3mm avalanche photodiodes (APD). Each telescope was interfaced with the APD using custom-designed coupling optics. The downward lidar APD and optics were connected to a logarithmic amplifier designed to increase the dynamic range. The data acquisition system consisted of two 100 Mhz 12-bit A/D cards with a Pentium 550Hz computer that controlled the laser interlock system, collected, stored and displayed the data in real time.

The data sets consisted of B measurements made continuously in a 2D planar region. One dimension was along the propagation axis, (the vertical) and the other was along the displacement of the aircraft, i.e. along horizontal straight paths at a fixed altitude of 4500m. The extent of the data sets in the horizontal was up to 120 km, while the spatial resolution in the horizontal was set by the aircraft speed and laser shot
averaging to 100 m. The extent of the data sets in the vertical was of the order of 4500 m and the spatial resolution was equal to the pulse length of 3 m. Therefore, the ratio of the largest to the smallest scales achieved was in the range 500-1000 and 1000-1500 in the horizontal and vertical respectively.

**Fig. 2:** The squares show the vertical structure function, the circles the horizontal for the ensemble of 9 vertical airborne lidar cross-sections. $\rho$ is the dimensionless backscatter ratio, the surrogate for the passive scalar aerosol density. $\Delta \rho$ is either the vertical or horizontal distance. The lines have the theoretical slopes $3/5$, $1/3$, they intersect at the spherio-scale here graphically estimated as 10cm. From [Lilley, 2003], [Lilley, et al., 2004].
Fig. 3a: Spectrum of $\rho$ as a function of horizontal wavenumber for the ensemble of 9 aircraft cross-sections described in the text. The line has the Kolmogorov slope (uncorrected for intermittency, -5/3). The regression slope is $-1.61$ from [Lilley, 2003], [Lilley, et al., 2004].
Fig. 3b: Spectrum of $\rho$ as a function of horizontal wavenumber for the ensemble of 9 aircraft cross-sections described in the text. The line has the Bolgiano-Obhukov slope (uncorrected for intermittency, $-11/5$). The regression slope is $-2.15$. From [Lilley, 2003], [Lilley, et al., 2004].

Analyzing the first order moment ($q=1$) case is interesting (fig. 2) because we expect $K(1)$ to be small enough that the horizontal and vertical $H$’s ($H_h$, and $H_v$) can be estimated as $\xi_h(1)=H_h=1/3$, $\xi_v(1)=H_v=3/5$. We can see from the figure that not only is the scaling excellent in both horizontal and vertical directions, but that in addition the exponents are very close to those expected theoretically. In fact, we find from linear regression: $H_h=0.33\pm0.03$, $H_v=0.60\pm0.04$. Also visible in the figure is the scale at which
the functions cross; this is a direct estimate of the sphero-scale which we find here varies between 2cm and 80cm, with an average of 10cm.

A standard method for the analysis of scaling and turbulent fields is the calculation of energy spectra. Here we calculate the 1-D spectra in both the horizontal and vertical directions; this is defined as the ensemble average of the modulus squared of the Fourier transform of the signal. Physically, \( E(k) \), “the energy at wavenumber \( k \)” gives the contribution to the total variance of the signal from structures between wavenumbers \( k \) and \( k + dk \). We have already noted that for scaling systems, \( E(k) = k^{-\beta} \); since \( E(k) \) is the Fourier transform of the autocorrelation, we have a simple relation between \( \beta \) and \( \xi(2) \):

\[
\beta = 1 + \xi(2) = 1 + 2H - K(2)
\]  

Equation (0.1)

Figures 3a,b shows the result. Again we find excellent scaling despite the slight increase at high wavenumber which is due to the presence of noise. Since the \( q=2 \) moments are more sensitive to large fluctuations (i.e., to higher order singularities) than \( q=1 \) statistics, the statistics are not quite as good. From the analysis below, we find \( K_h(2) = 0.065 \), \( K_v(2) = 0.10 \), hence the theoretical spectral exponents are: \( \beta_h = 1.60 \pm 0.03 \), \( \beta_v = 2.10 \pm 0.04 \); these lines are drawn on the figures for reference; these are within one standard deviation of the regression values \( \beta_h = 1.61 \pm 0.03 \), \( \beta_v = 2.15 \pm 0.04 \).

In figures 2, 3, it is important to emphasize that these first order structure functions and (second order) spectra were not fit to the data; the theoretical reference slopes are drawn. These first simultaneous measurements on atmospheric cross-sections permitted [Lilley, et al., 2004] the elliptical dimension \( D_s \) to be estimated as 2.55±0.02,
clearly eliminating the contending 2-D theory or leading gravity wave theory (which have $D_S=2, 7/3$ respectively).

The above first and second order statistics are only very partial descriptions of the fields. In order to more completely test the anisotropic $23/9D$ multifractal model discussed in part I, we must investigate the statistics of all orders, i.e. including the intermittency. In particular, we are interested in testing the hypothesis a) that the passive scalar field is a universal multifractal, and that the lower and higher order statistics (which correspond to weak or strong structures/events) are stratified in the same way as the mean (and variance) fields investigated in figs 2, 3.

3. DIRECT TEST OF THE 23/9D MODEL USING ATMOSPHERIC AEROSOLS AND LIDAR DATA:

3.1 The statistics of passive scalar advection:

3.1.1 The Anisotropic Corrsin-Obukov law:

In optically thin media the backscatter ratio is a good surrogate for the aerosol concentration so that if one ignores sources or sinks of aerosols and chemical reactions occurring during the roughly 20 minutes during which the data were acquired, it will be an approximation to a passively advected tracer (“scalar”) (with or without wavelike fractional integration). We now consider the predictions of the $23/9D$ model for such passive scalars. By introducing the scale function, the $23/9D$ model automatically predicts anisotropic generalizations of many of the standard results of isotropic turbulence theory, including the standard [Corrsin, 1951], [Obukhov, 1949] theory of passive scalar advection. The standard isotropic theory is based on two quadratic
invariants: the energy flux for the wind field (see section 2.2.3) and the passive scalar variance flux $\chi$ so that:

$$\Delta \rho(\Delta r) = \chi^{1/2}_\Delta e^{-1/6} |\Delta r|^{1/3}; \quad \Delta \rho(\Delta r) = \rho(r + \Delta r) - \rho(r)$$  \hspace{1cm} (0.2)

The subscripts indicate the spatial resolutions of the fluxes. As discussed in part I, in order to obtain the anisotropic generalization, we need only make the replacement $|\Delta r| \to \|\Delta r\|$ (we consider only space here). By taking $\Delta r=(\Delta x,0,0)$ and $\Delta r=(0,0,\Delta z)$, this yields the following horizontal and vertical laws:

$$\Delta \rho(\Delta x) = \chi^{1/2}_\Delta e^{-1/6} \Delta x^{1/3}$$
$$\Delta \rho(\Delta z) = \chi^{1/2}_\Delta e^{-1/2} \phi^{1/5}_\Delta \Delta z^{3/5}$$  \hspace{1cm} (0.3)

The first is the standard Corrsin-Obukhov law while the second is new.

Although the lidar only measures a surrogate for $\rho$, according the the 23/9 D model, any physical atmospheric field whose dynamics are controlled by the fluxes $\varepsilon$ and $\phi$ should have the same scale function and hence the same ratio of horizontal to vertical exponents. Hence, the experiment can still estimate $H_z$ and hence $D_z$ even if the relation between $B$ and $\rho$ is nonlinear or is only statistical in nature.

### 3.1.2 The statistical moments:

Up until now, we have ignored intermittency, concentrating instead on the predictions of spatially homogeneous turbulence theories. However, during the 1980’s it became increasingly recognized that turbulent scaling regimes often had cascade phenomenologies generically leading to strong multifractal intermittency. For example,
taking $q^{th}$ powers of eq. 0.2 and performing ensemble averaging, we expect the following statistics in passive scalar advection:

$$\left\langle |\Delta \rho(\Delta r)|^q \right\rangle = \left\langle \varphi_{|\Delta r|}^q \right\rangle \|\Delta r\|^{q/3}; \quad \varphi_{|\Delta r|} = \chi_{|\Delta r|}^1 \chi_{|\Delta r|}^{1/6}$$

(0.4)

By introducing the structure function exponent $\xi(q)$, and the $\phi$ flux exponent $K(q)$, we may write this as:

$$\left\langle |\Delta \rho(\Delta r)|^q \right\rangle \propto \|\Delta r\|^{\xi(q)}$$

(0.5)

$$\left\langle \varphi_{|\Delta r|}^q \right\rangle \propto \|\Delta r\|^{-K(q)}$$

(0.6)

with:

$$\xi(q) = qH - K(q); \quad H = 1/3;$$

(0.7)

In the general cascade theory, the only restriction of $K(q)$ is that it is convex. However, due to the existence of stable, attractive multifractal universality classes (the multiplicative analogue of the additive central limit theorem in probability theory see Schertzer and Lovejoy 1987, 1997), under fairly general circumstances, $K(q)$ is determined by two basic parameters as:

$$K(q) = \frac{C_1}{\alpha - 1} (q^\alpha - q)$$

(0.8)

where $C_1$ is the codimension characterizing the sparseness of the mean field whereas $0 \leq \alpha \leq 2$ is the index of multifractality (the Levy index of the generator); it characterizes the relative importance of low “holes” in the field ($\alpha=0$ zero totally hole dominated, it is the monofractal limiting case).

As usual, from the statistics expressed in terms of the scale function, we can recover separate horizontal and vertical statistics by using $\Delta r=(\Delta x,0,0)$ and $\Delta r=(0,0,\Delta z)$.
respectively. We see that the ratio of horizontal to vertical exponents is expected to be \( H_z \).

### 3.2 Multifractal analysis:

#### 3.2.1 \( q \)th order Structure functions:

In order to fully test the theory, we need to compare horizontal and vertical \( \xi(q) \) and \( K(q) \) exponents. The simplest way is to calculate the structure functions which are simply the moments of the absolute differences (see eq. 0.2); this is a “poor man’s wavelet”, adequate for our purposes. Fig. 4a,b shows the scaling in the horizontal and vertical for the structure functions of order 0 to 5 estimates and fig. 5a,b shows the corresponding exponents \( \xi_h(q) \), \( \xi_v(q) \) obtained from the slopes. The straight lines \( qH \) are shown; the deviations are purely due to the multifractal intermittency corrections \( K(q) \) which we study in the next subsection.
**Fig. 4a:** Horizontal structure functions of order $q$ between 0 and 5 at increments of 0.5, the regressions were estimated over the scaling range.
Fig. 4b: Vertical structure functions of order $q$ between 0 and 5 at increments of 0.5, the regressions were estimated over the scaling range.
Fig. 5a: The scaling exponent $\xi(q)$ of the $q^{th}$ order structure functions; the slopes of fig. 4a.
Fig. 5b: The scaling exponent $\xi(q)$ of the $q^{th}$ order structure functions; the slopes of fig. 4b.

3.2.2 Trace Moments, and $C_1$:

In order to characterize $\xi(q)$ we need to estimate the nonlinear part, $K(q)$. However, due to the fact that the $C_1$'s are much smaller than the H's we find that for low $q$, $K(q)$ will be much smaller than $\xi(q)$. It is therefore best to estimate $K(q)$ directly, this can be achieved by removing the linear scaling $\|\Delta x\|^H$ in eq. 0.4 so as to study the scaling of the fluxes directly. This can be achieved by fractionally differentiating $\rho$ by $H_h$ in the horizontal, and by $H_v$ in the vertical (this is simply a Fourier filter of $k^{Hv}$). We
have mentioned (eq. 20) that the basic characterization of $K(q)$ is the sparseness of the mean ($C_1$) and the multifractal index $\alpha$ (see fig. 6a, b). Using moments near unity allows us to use the relation $C_1 = K'(1)$, to estimate $C_1$, by numerically differentiating $K(q)$. We obtain $C_{1,h} = 0.037 \pm 0.006$, $C_{1,v} = 0.059 \pm 0.007$. From eq. 0.8 we see that if the universality hypothesis holds (see next subsection), the ratio $C_{1,h} / C_{1,v} = H_z$ and $\alpha_h = \alpha_v$. In fig. 7 we show the corresponding scatter plot with the $H$ values. Due to their much smaller values, the relative spread in the $C_1$ values is larger than for $H$. However, when the standard error estimates are included, $C_{1,h} / C_{1,v} = 0.70 \pm 0.2$ is compatible with the value $5/9$.

![Fig. 6a: The scaling exponent function $K(q)$ for the horizontal direction (from the slopes of moments such as those in fig. 5a). The curve is the universal multifractal with parameters $C_{1h} = 0.037$, $\alpha_h = 1.82$.](image-url)
Fig. 6b: The scaling exponent function $K(q)$ for the horizontal direction (from the slopes of moments such as those in fig. 5b). The curve is the universal multifractal with parameters $C_1=0.059$, $\alpha_1=1.83$. 

![Graph showing the scaling exponent function $K_q(q)$ for the horizontal direction, with a universal multifractal curve and parameters $C_1=0.059$, $\alpha_1=1.83$.](image)
Fig. 7: A scatterplot of the basic universal multifractal exponents $C_1$, $H$ as estimated from trace moments and structure functions respectively for each of the 9 cross-sections. Shown for reference is the theoretical slope $H_z=5/9$. The scatter is within a standard error; see the text. This shows that the strong (intermittent) structures are also stratified with the same exponent $H_z$. 
Fig. 8a: The horizontal $K(q, \eta)$ as a function of $\eta$; the regression lines have slopes $\alpha_h = 1.82$. Each line has a different $q$ value, form top to bottom $q = ?$ (Marc, I couldn’t find this info?).
3.3.3 Double Trace Moments, and $\alpha$:

In principle, we could perform a nonlinear regression in $K(q)$ to determine $\alpha$ as well as $C_1$. In practice however, the regression is not very well posed; this is particularly true since the universal form (eq. 15??) is only valid for $q$’s below a critical value after which $K(q)$ becomes linear. This “multifractal phase transition” arises because either the sample size is too small to estimate the high order moments, or because of the divergence of moments greater than a critical value $q_D$. A better way to estimate the value of $\alpha$ is via the double trace moment technique. As for the trace moment method, one starts with the
fluxes (at the finest resolution \( \Lambda \)) after fractionally differentiating \( \rho \). One then raises the flux to the \( \eta \) power and degrades the resolution to an intermediate resolution \( \lambda \):

\[
\left\langle \left( \varphi^{\eta}_{\lambda} \right)^\eta \right\rangle = \lambda^{K(q,\eta)}
\]

(0.9)

The exponent \( K(q,\eta) \) is related to \( K(q) \) via:

\[
K(q,\eta) = K(q\eta) - qK(\eta)
\]

(0.10)

so that if \( K(q) = K(q,1) \) is of the universal form (eq. 0.8), then we have the particularly simple relation:

\[
K(q,\eta) = \eta^\alpha K(q,1) = \eta^\alpha K(q)
\]

(0.11)

so that for fixed \( q \), \( \alpha \) can be determined directly by log-log regression of \( K(q,\eta) \) versus \( \eta \).

Fig. 8a,b shows the results for \( K(q,\eta) \) in the horizontal and vertical respectively. The linearity shows that the universality hypothesis is accurately obeyed. From the regressions, we obtain: \( \alpha_h = 1.82 \pm 0.05 \), \( \alpha_v = 1.83 \pm 0.04 \). As predicted by the scaling anisotropy hypothesis, these are equal within error bars. Finally, from the measured values of \( \alpha \) and the regression intercepts \( K(q,1) \), we obtain the additional estimates \( C_{1h} = 0.037 \pm 0.006 \); \( C_{1v} = 0.059 \pm 0.007 \) which are very close to those obtained from the trace moment method discussed above.

### 3.3.4 The cross-section to cross-section variability:

Up until now, we have mostly pooled the data from the 9 cross-sections in order to obtain improved statistics. However, it is of interest to confirm that the statistics for individual cross-sections are indeed close to each other, for example, that they are not from totally different statistical ensembles. Also, since the spherio-scale depends on two
highly variable fluxes, we anticipate that it will vary considerably about the ensemble estimate 10cm. In table 2 we give the values of $l_s$ and $\langle \Delta \rho(l_s) \rangle$; we notice a slight tendency for the larger $l_s$ cases (less stratification) to occur for when $\langle \Delta \rho(l_s) \rangle$ is larger, overall $l_s$ varies from 2cm to 80cm. Also, in table 2 we see the cross-section to cross-section variation of the universal multifractal parameters; it is generally small.

<table>
<thead>
<tr>
<th>Dataset</th>
<th>$H_h$</th>
<th>$H_v$</th>
<th>$H_z$</th>
<th>$\alpha_h$</th>
<th>$\alpha_v$</th>
<th>$C_{1,h}$</th>
<th>$C_{1,v}$</th>
<th>$C_{1,v}$</th>
<th>$\Delta \rho(l_s)$</th>
<th>$l_s$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>08-14-t5</td>
<td>0.35</td>
<td>0.62</td>
<td>0.56</td>
<td>1.86</td>
<td>1.85</td>
<td>0.031</td>
<td>0.064</td>
<td>0.48</td>
<td>0.018</td>
<td>0.03</td>
</tr>
<tr>
<td>08-14-t7</td>
<td>0.36</td>
<td>0.63</td>
<td>0.57</td>
<td>1.82</td>
<td>1.78</td>
<td>0.027</td>
<td>0.048</td>
<td>0.56</td>
<td>0.025</td>
<td>0.03</td>
</tr>
<tr>
<td>08-14-t17</td>
<td>0.33</td>
<td>0.59</td>
<td>0.56</td>
<td>1.90</td>
<td>1.90</td>
<td>0.029</td>
<td>0.059</td>
<td>0.49</td>
<td>0.050</td>
<td>0.79</td>
</tr>
<tr>
<td>08-14-t20</td>
<td>0.34</td>
<td>0.60</td>
<td>0.56</td>
<td>1.90</td>
<td>1.85</td>
<td>0.044</td>
<td>0.049</td>
<td>0.89</td>
<td>0.056</td>
<td>0.63</td>
</tr>
<tr>
<td>08-15-t20</td>
<td>0.31</td>
<td>0.61</td>
<td>0.51</td>
<td>1.80</td>
<td>1.80</td>
<td>0.040</td>
<td>0.039</td>
<td>1.02</td>
<td>0.022</td>
<td>0.1</td>
</tr>
<tr>
<td>08-15-t2</td>
<td>0.33</td>
<td>0.60</td>
<td>0.55</td>
<td>1.77</td>
<td>1.81</td>
<td>0.037</td>
<td>0.052</td>
<td>0.71</td>
<td>0.020</td>
<td>0.079</td>
</tr>
<tr>
<td>08-15-t6</td>
<td>0.39</td>
<td>0.69</td>
<td>0.56</td>
<td>1.87</td>
<td>1.80</td>
<td>0.039</td>
<td>0.059</td>
<td>0.66</td>
<td>0.063</td>
<td>0.31</td>
</tr>
<tr>
<td>08-15-t8</td>
<td>0.38</td>
<td>0.65</td>
<td>0.58</td>
<td>1.76</td>
<td>1.80</td>
<td>0.040</td>
<td>0.050</td>
<td>0.80</td>
<td>0.036</td>
<td>0.1</td>
</tr>
<tr>
<td>08-15-t22</td>
<td>0.32</td>
<td>0.59</td>
<td>0.54</td>
<td>1.85</td>
<td>1.85</td>
<td>0.037</td>
<td>0.051</td>
<td>0.72</td>
<td>0.045</td>
<td>0.31</td>
</tr>
<tr>
<td>Ensemble</td>
<td><strong>0.33</strong></td>
<td><strong>0.60</strong></td>
<td><strong>0.55</strong></td>
<td><strong>1.82</strong></td>
<td><strong>1.83</strong></td>
<td><strong>0.037</strong></td>
<td><strong>0.053</strong></td>
<td><strong>0.72</strong></td>
<td>-</td>
<td><strong>0.10</strong></td>
</tr>
<tr>
<td>Error</td>
<td>±0.03</td>
<td>±0.04</td>
<td>±0.02</td>
<td>±0.05</td>
<td>±0.04</td>
<td>±0.006</td>
<td>±0.007</td>
<td>±0.2</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

**Table 2:** A comparison of various universal multifractal parameters as estimated for each of the 9 cross-sections.

Overall we find that $H_h$ varies between 0.31 and 0.39 with an ensemble average mean of 0.33±0.03 while $H_v$ varies between 0.59 and 0.69 with and ensemble mean value of 0.60±0.04 (note that the values quotes in the row “ensemble” are not the averages of the
values for the individual datasets, they are the values found from regression for the actual ensemble statistics). \( H_z \) varies between 0.51 and 0.58 with an ensemble mean of 0.55±0.02. Similar comparisons can be done for the other parameters.

3.4. **Comparison with other multifractal results on passive scalars:**

It is interesting to compare our parameter estimates with those of other passive scalars in the literature. Table 3 displays a number of other results. Caution should be used in this comparison, since with only one exception, the literature values are for variations in time whereas we analyse (nearly) pure spatial data. Since as discussed in part I there is a space-time anisotropy (if we ignore the effect of horizontal and vertical wind it is characterized by \( H_t = 2/3 \) in the place of \( H_z \)) we should expect \( C_1, H \) to differ by factor \( H_t \). However, as we discuss elsewhere ([Radkevitch, et al., 2005]), the time variation is often dominated by advection in which case we expect \( C_1, H \) to have the horizontal values. From the table, we find that while our \( \alpha \) values are generally a little higher, those of \( C_1 \) are considerably higher. This may be a consequence of the fact that the lidar measured concentration surrogate is actually nonlinearly related to the measured \( B \); from eq. 26, if \( \rho = B^\eta \), then we have \( C_1 \rho = C_1 B^\eta \); this is discussed in [Lilley, et al., 2004].
### Table 3:

<table>
<thead>
<tr>
<th>Reference</th>
<th>Field</th>
<th>Type</th>
<th>$\alpha$</th>
<th>$C_1$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Finn, et al., 2001]</td>
<td>$S_{f_6}$</td>
<td>Time</td>
<td>1.65</td>
<td>0.11</td>
<td>0.40</td>
</tr>
<tr>
<td>[Finn, et al., 2001]</td>
<td>$H_{20}$</td>
<td>Time</td>
<td>1.60</td>
<td>0.07</td>
<td>-</td>
</tr>
<tr>
<td>[Finn, et al., 2001]</td>
<td>$T$</td>
<td>Time</td>
<td>1.69</td>
<td>0.09</td>
<td>0.44</td>
</tr>
<tr>
<td>[Schmitt, et al., 1996]</td>
<td>$T$</td>
<td>Time</td>
<td>1.45</td>
<td>0.07</td>
<td>0.38</td>
</tr>
<tr>
<td>[Pelletier, 1995]</td>
<td>$T$</td>
<td>Time</td>
<td>1.69</td>
<td>0.08</td>
<td>-</td>
</tr>
<tr>
<td>[Pelletier, 1995]</td>
<td>$H_{20}$</td>
<td>Time</td>
<td>1.69</td>
<td>0.08</td>
<td>-</td>
</tr>
<tr>
<td>[Wang, 1995]</td>
<td>$T$</td>
<td>Time</td>
<td>1.69</td>
<td>0.10</td>
<td>0.41</td>
</tr>
<tr>
<td>[Chigirinskaya, et al., 1994]</td>
<td>$T$</td>
<td>Space</td>
<td>1.25</td>
<td>0.04</td>
<td>0.33</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Horizontal)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[Schmitt, et al., 1992]</td>
<td>$T$</td>
<td>Time</td>
<td>1.4</td>
<td>0.22</td>
<td>0.33</td>
</tr>
<tr>
<td>Average</td>
<td>Scalar</td>
<td>Time</td>
<td>1.64</td>
<td>0.085</td>
<td>0.41</td>
</tr>
</tbody>
</table>

This table summarizes the results of various experiments which obtained estimates of universal multifractal parameters for turbulent passive scalars in the atmosphere. Clearly it is possible to make a strict and direct comparison between the results in this table and ours. In addition, more the majority of the results were for temperature which which is not obviously passive at all! Despite these limitations, there is fairly good quantitative agreement between the values obtained in the earlier studies and the values reported here.

### 3.5 Analysis of the anisotropy:

In multifractals there is a one to one correspondence between singularities (intensity levels) and statistical moments, hence by examining the stratification of both high and low order statistical moments, we are in fact determining whether both intense and weak structures are differentially stratified to the same degree (they have the same $H_z$). In its simplest version presented here, there is a single scale function the same for high and low order moments so that the $23/9D$ model predicts scaling anisotropy for all the exponents: $\xi_h(q)/\xi_v(q)=K_h(q)/K_v(q)=H_z$. We have seen that in terms of the universal...
mulitfractal parameters, this implies $H_h/H_v=C_{1h}/C_{1v}=H_z$ and $\alpha_h=\alpha_v$, and that these relations are at least approximately verified. However, this confirmation is somewhat indirect since it relies on both the universality hypothesis and also on parameter fitting. It is therefore of interest to test the ratios directly. In fig. 9 we do this for $\xi(q)$. We see that at least for $q<2$ (where the moments are accurately measured), the data are close to the theoretical line slope $H_z$.

![Fig. 9: Scatterplots of $\xi_h(q)$ and $\xi_v(q)$ obtained for each cross-section using the trace moment method (X’s) after adding $qH$, and directly from the structure functions (+’s). The reference line has slope $H_z=5/9$.](image)
4. CONCLUSIONS

One of the most basic aspects of atmospheric structure is its stratification. In part I we discussed various models and proposed a new one – a gravity/turbulence wave generalization of the classical 23/9D model in which the stratification is differential, i.e. the typical “flatness” or anistropy of structures increases with scale in a scaling way i.e. without characteristic length scale. In this part II, we considered the experimental evidence, first reviewing the data on horizontal and vertical statistics; we argued that they were compatible with the value $D_s = 23/9$ rather than 2 or 7/3 (the competing 2D and linear gravity wave theories respectively). However, the only direct way to investigate the stratification is through vertical cross-sections. With the advent of high powered lidars with logarithmic amplifiers this is now possible. Here we studied stratified structures spanning over three orders of magnitudes in both horizontal and vertical scales. Using such state-of-the-art lidar data [Lilley, et al., 2004] made the first direct measurements of the elliptical dimension $D_s$ characterizing the stratification finding that it is $D_s = 2.55 \pm 0.02$ which is very close to the theoretically predicted value $23/9 = 2.555…$ but quite far from the standard values 2 (completely flat) or 3 (completely isotropic). In this paper, we extend the [Lilley, et al., 2004] study by examining the stratification of both high and low order statistical moments, we showed that both intense and weak structures were apparently differentially stratified to the same degree (same $H_z$).

The “unified scaling” or “23/9D” theory which predicts this result is based on the primacy of buoyancy forces in determining the vertical structure while allowing energy fluxes to determine the horizontal structure. It predicts the observed wide range scaling in cloud radiances, and – as our review shows – it is compatible with the available
observations of both the horizontal and vertical wind and temperature spectra. In contrast, the standard model does not directly consider the buoyancy at all and it involves two isotropic regimes – at small scales it is 3D energy driven while at large scales it is 2D and both enstrophy and energy driven.

The 23/9D turbulent model is physically satisfying since it finally allows buoyancy to play the role of fundamental driver of the dynamics. With the allowance for a wavelike fractional integration, it can be compatible with gravity wave-like phenomenology. While to numerical weather forecasters this may seem academic, up until now virtually all turbulent theories have been energy or enstrophy driven. The model also explains the difficulty in making aircraft measurements of horizontal structure: 23/9D turbulence can lead to fractal aircraft trajectories i.e. long range biases so that the spectra may be incorrectly interpreted. In addition, a very small average vertical gradient leads to a transition from $k^{-5/3}$ to $k^{-11/5}$; we quantitively showed this on the two major campaigns to date: GASP and MOZAIC. Finally, the 23/9D model naturally explains how the horizontal structures in the atmosphere can display wide range scaling, right through the meso-scale.

Although the implications of the model may take many years to discern, an immediate implication is for the treatment of stratification in numerical weather and climate models. Boussinesq, hydrostatic and related approximations require scale separations and are not compatible with the observations.

5. REFERENCES:


Van der Hoven, I. (1957), Power spectrum of horizontal wind speed in the frequency range from 0.0007 to 900 cycles per hour, *Journal of Meteorology*, 14, 160-164.

