Let \( s(n) \equiv n(x) - \langle n \rangle \)
so it the variance of the density from the average number density. As above the intensity of scattered radiation etc. is
\[
\hat{I}(q) \propto \hat{S}(q)
\]
where \( \hat{S}(q) \) is the structure factor, and
\[
\langle \hat{S}(k) \hat{S}(k') \rangle = \hat{S}(k) (2\pi)^3 \delta (k + k'),
\]
and
\[
\hat{S}(k) = \int d^3 \nu \ e^{-ik \cdot \nu} \langle s(x) s(0) \rangle
\]
using translational invariance of space. On recent problem set, one question was to prove
\[
\langle (\Delta n)^2 \rangle = \langle s^2 \rangle = n^2 k_B T n^2 k_B T,
\]
where \( n \) is the number density and \( k_B T \) is the isothermal compressibility. This relationship between fluctuations and a thermodynamic derivative is very common. This is only one example.
This is the fluctuation in a volume $V$. Clearly
\[ \langle \mathbf{S} \rangle = \frac{1}{V} \int \frac{d^3 \mathbf{r}}{V} \mathbf{S}(\mathbf{r}). \]
over a volume $V$ and
\[ \langle \mathbf{S} \rangle^2 = \frac{1}{V} \int \frac{d^3 \mathbf{r}}{V} \int \frac{d^3 \mathbf{r}'}{V} \langle \mathbf{S}(\mathbf{r}) \mathbf{S}(\mathbf{r}') \rangle = \frac{n^2 k_B T}{V} g \rho_T^2. \]
Using translational invariance again, we have
\[ \int \frac{d^3 \mathbf{r}}{V} \langle \mathbf{S}(\mathbf{r}) \mathbf{S}(\mathbf{r}) \rangle = \frac{n^2 k_B T}{V} g \rho_T^2. \]
So \[ \langle \mathbf{S}(\mathbf{r}) \mathbf{S}(\mathbf{r}) \rangle \text{ area under curve gives } \frac{n^2 k_B T}{V} g \rho_T^2. \]

(This is a little rough, I should use rotational invariance and then \[ \int d^2 r = 2 \pi \int_0^\infty \! r dr \] and \[ 2 \pi \int_0^\infty \! r dr \langle \mathbf{S}(\mathbf{r}) \mathbf{S}(\mathbf{r}) \rangle = \frac{n^2 k_B T}{V} g \rho_T^2. \])

Usually this is expressed as the "sum rule"
\[ \lim_{k \to 0} \hat{S}(k) = \frac{n^2 k_B T}{V} g \rho_T^2. \]
So, say the scattering looks like this,

\[ S(k) = \frac{-i k \cdot n}{\hbar^2 k T} \]

with structure on small length scales.

We need a Fourier transform cookbook to do field theory.

\[ S^\ast(n) = \int d^3r \ e^{-i k \cdot n} S(n) \]

Note \( S^\ast(k) = S(-k) \) because \( S \) is real.

\[ S(n) = \begin{cases} \frac{1}{V} \sum_k e^{i k \cdot n} S(k) & \text{DISCRETE} \\ \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot n} S(k) & \text{CONTINUOUS} \end{cases} \]

\[ S(k) = \int \frac{d^3r}{(2\pi)^3} e^{-i k \cdot n} \]

\[ S(n) = \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot n} \]

where \( \int d^3r \ f(n) \ S(n-v_0) = f(v_0) \).

The Kronecker delta function is

\[ S_{k,0} = \frac{1}{V} \int d^3r \ e^{i k \cdot n} = \begin{cases} -1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \]
the Dirac delta function also has
the representation
\[ \delta(\mathbf{r}) = \frac{1}{V} \sum_{k} e^{i \mathbf{k} \cdot \mathbf{r}} \]
Note that moving back and forth between
the continuous and discrete forms makes
use of the density of states in \( k \)
space as \( \frac{V}{(2\pi)^3} \), as we used
much earlier in the course. So
\[ \sum_{k} \leftrightarrow \frac{V}{(2\pi)^3} \int d^3k \]
\[ \delta_{k,0} \leftrightarrow \frac{V}{(2\pi)^3} \delta(k) \]
It is handy to go back and forth
between the continuous and discrete
representations.

In condensed matter physics we can
think of the \( \sum_{k} \) as the
fundamental quantity and \( \frac{V}{(2\pi)^3} \int d^3k \) as
the approximation. The smallest length
scale in condensed matter is, e.g. the size of an atom, \( a \). The largest length scale is the size of the system \( L = (V)^{1/3} \).

Since we are interested in the thermodynamic limit of \( L \rightarrow \infty \), we can leave \( a \) fixed and small. In wavenumbers, our integrals are \( \left( \frac{2\pi}{L} \rightarrow 0 \right) < k < \frac{2\pi}{a} \).

So we are concerned with the "infra-red" limit of \( k \rightarrow 0 \). In particle physics, the concern is with large energies, the "ultra-violet" limit of \( k \rightarrow \infty \). This is an aside.

Of course we need something interesting to evaluate, an energy and a system of energy states. In particular, if \( S \) is small - we have

\[
E_{\text{state}} = \int d\mathbf{r} \left[ \frac{A}{z^2} S^2 \right]
\]

to lowest order where \( A \) is a constant.
independent of \( G \) (we will show it is related to \( \mu - 1 \)). Note a linear term in \( G \) could be transformed away. This form obviously does not have any interesting properties for something like \( \langle \delta(r) \delta(0) \rangle \), as \( E_{\text{state}} \) is the same for all \( G(\mu) \). Imagine, like our development of the Ising model, that the system "likes" to have local values of the \( \delta(x) \) close to each other. That is, the energy has a term of the form

\[
\left( \sum _{\in \text{neighbourhood of } r} (\delta(x) - \delta(x')) \right)^2
\]

Clearly, expanding this gives a term of the form

\[
(\nabla \delta(x))^2
\]

and our energy has the form:
\[ E_{\text{STATE}} = \int d^3v \left[ \frac{A}{2} \sqrt{s}^2 + \frac{B}{2} (\nabla \sqrt{s})^2 \right] \]

(when \( B \) is a constant)
to lowest order in \( s \), and lowest order in spatial variations of \( s \). This follows from such general and simple arguments that these two terms almost always appear in the energetics of field theories. In particular, the continuum Ising model has the form

\[ \rightarrow \int d^4v \left[ \frac{A}{2} \sqrt{s}^2 + \frac{B}{2} (\nabla \sqrt{s})^2 + \frac{C}{4} \sqrt{s}^4 \right] \]

Our partition function is

\[ Z_1 = \left( \frac{1}{\sqrt{\pi}} \right) \int d\sqrt{s}(u) e^{-E_{\text{STATE}}} \]

where our previous \( \prod \Sigma_i \rightarrow \int d\sqrt{s}(u) \) in

continuous notation. More or less by inspection we can see that \( Z_1 \) involves many Gaussians coupled to each other through the \( \int d\nu \frac{B}{2} (\nabla \sqrt{s})^2 \) term. We will disentangle all those
Gaussians with Fourier transforms and eventually get; as we already know,

\[ \langle \hat{\psi}(k) \hat{\psi}^*(k') \rangle = \hat{\mathcal{S}}(k)(2\pi)^3 \delta(k + k') \]

with \( \hat{\mathcal{S}}(k) = \frac{k_B T}{A + B k^2} \)

\[ = \frac{n^2 k_B T g L T}{1 + \bar{\xi}^2 k^2} \]

where \( A = \frac{1}{n^2 q T} \) and \( \bar{\xi}^2 = B/A \).

For large \( r \) we obtain the correlation function

\[ \langle \phi(r) \phi(0) \rangle = \int d^3 r e^{-i k \cdot r} \hat{\mathcal{S}}(k) \]

\[ \approx e^{-r/\bar{\xi}} \]

It is not difficult, but it is a little tedious.

The gradient term becomes simple in Fourier space

\[ \int d^3 r (\nabla \phi)^2 = \int d^2 \mathbf{r} \left( \frac{\partial}{\partial \mathbf{r}} \int \frac{d^3 k e^{i k \cdot r}}{(2\pi)^3} \hat{\mathcal{S}}(k) \right)^2 \]
\[ \int d^3 r (\nabla \bar{\Psi})^2 = \int d^2 r \left( - \int \frac{d^3 k}{(2\pi)^3} e^{i \mathbf{k} \cdot \mathbf{r}} \bar{\Psi}(\mathbf{k}) \mathbf{k} \cdot \int \frac{d^3 k'}{(2\pi)^3} e^{-i \mathbf{k}' \cdot \mathbf{r}} \bar{\Psi}(\mathbf{k'}) \right) \]

Doing the integral over \( r \) gives

= \left\{ \frac{1}{(2\pi)^3} \int \frac{d^3 k}{(2\pi)^3} e^{i (\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}} \bar{\Psi}(\mathbf{k}) \bar{\Psi}(\mathbf{k'}) \mathbf{k} \cdot \mathbf{k'} \int \frac{d^3 r}{(2\pi)^3} e^{-i (\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}} \bar{\Psi}(\mathbf{k} + \mathbf{k'}) \right\}

and

\[ \int d^3 r (\nabla \bar{\Psi})^2 = \int \frac{d^3 k}{(2\pi)^3} k^2 \bar{\Psi}(\mathbf{k}) \bar{\Psi}(-\mathbf{k}) \]

= \left\{ \int \frac{d^3 k}{(2\pi)^3} k^2 \left| \bar{\Psi}(\mathbf{k}) \right|^2 \right\}

= \int \frac{1}{(2\pi)^3} \sum_{\mathbf{k}} \left| \bar{\Psi}(\mathbf{k}) \right|^2

Since \( \bar{\Psi}^*(\mathbf{k}) = \bar{\Psi}(-\mathbf{k}) \), this turns differentiation into multiplication by \( \mathbf{k} \), and essentially diagonalizes the matrix \[ [-\nabla^2 \delta(\mathbf{x} - \mathbf{y})] \]

Now

\[ \sum_{\text{states}} = \prod_{\mathbf{k}} \int d\bar{\Psi}(\mathbf{k}) = \prod_{\mathbf{k}} d^2 \bar{\Psi}(\mathbf{k}) \]

where \( d^2 \bar{\Psi}(\mathbf{k}) = d \text{Re} \bar{\Psi}(\mathbf{k}) d \text{Im} \bar{\Psi}(\mathbf{k}) \)

and the prime on the product is a
reminder that we must restrict the k's to half of k space to avoid double counting (because \( \Sigma(k) \) is real).

For example, restrict to shaded region

If we were in 2 dimensions. Using the energy after Fourier transforming the probability weight for a state (up to a proportionality constant)

\[
\text{Probability } \propto \exp - \frac{1}{2 \text{k_B} T \text{V}} \sum_{k} (A + B k^2) |\hat{S}(k)|^2
\]

Hence

\[
\langle |\hat{S}(\mathbf{q})|^2 \rangle = \prod_{k} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} |\hat{S}(\mathbf{q})|^2 \exp - \frac{1}{2 \text{k_B} T \text{V}} \sum_{k} (A + B k^2) |\hat{S}(k)|^2
\]

Almost all the terms cancel out, except the one we are interested in. But there is still a subtlety involving...
Consider the term which involves $\hat{S}_{42}$, in

$$\sum_{k \neq k_0} k^2 |\hat{S}_k|^2.$$ This term is

$$(42)^2 \hat{S}_{42} \hat{S}_{-42} + (42)^2 \hat{S}_{-42} \hat{S}_{42} = 2 \frac{(42)^2 \hat{S}_{42} \hat{S}_{-42} - \hat{S}_{42} \hat{S}_{42}}{42}$$

so a factor of $2$ appears and we have

$$\langle |\hat{S}(q)|^2 \rangle = \frac{\int d^2 \hat{S}(q) |\hat{S}(q)|^2 \exp \left( \frac{1}{k_0 V} (A + Bq^2) |\hat{S}(q)|^2 \right)}{\int d^2 \hat{S}(q) \exp \left( \frac{1}{k_0 V} (A + Bq^2) |\hat{S}(q)|^2 \right)}$$

we write $\hat{S}(q) = \text{Re} \ i^{\alpha}$, then

$$\langle |\hat{S}(q)|^2 \rangle = \int_0^\infty R dR R^2 e^{-R^2/a^2}$$

where $a = \frac{k_0}{k_{31} V}$. $A + Bq^2$ and we can easily obtain

$$\langle |\hat{S}(q)|^2 \rangle = \frac{k_{31} T}{A + Bq^2} V.$$
and so
\[ \langle \hat{\mathbf{S}}(\mathbf{k}) \hat{\mathbf{S}}(\mathbf{k}') \rangle = \begin{cases} \frac{k_{\text{v}}}{A + B k^2} & \text{DISCRETE} \\
\frac{k_{\text{v}}}{A + B k^2} \left(2\pi\right)^3 \delta(\mathbf{k} + \mathbf{k}') & \text{CONTINUOUS} \end{cases} \]

Hence
\[ \hat{\mathbf{S}}(\mathbf{k}) = \frac{k_{\text{v}}}{A + B k^2} \]

and since \( \hat{\mathbf{S}}(\mathbf{0}) = n^2 k_{\text{v}} T \Theta T \), we have
\[ A = \frac{1}{n^2 k_{\text{v}}} \]

where \( \xi^2 = B / A \), and the scattering

obeys \( \hat{\mathbf{S}}(\mathbf{k}) \) Lorentzian, close to \( \mathbf{k} = \mathbf{0} \).

Our theory is good for small \((\nabla \mathbf{S})^2 \)
by construction, so that corresponds to
small \( k \), \((\nabla \mathbf{S})^2 \leftrightarrow k^2 |\hat{\mathbf{S}}_{\mathbf{k}}|^2 \).
In real space our result for \( S(k) \) can be Fourier transformed. We can use the green function \( g(r) \) for the Helmholtz function

\[
\nabla^2 g(r) - \xi^{-2} g(r) = \delta(r)
\]

\[
\hat{g}(k) = \frac{1}{k^2 + \xi^{-2}}
\]

and, in three dimensions,

\[
g(r) = \frac{e^{-r/\xi}}{4\pi r}
\]

so

\[
\langle S(r) S(0) \rangle = \frac{\hbar^2 k_B T}{4\pi^2 \xi^2 r} \frac{3}{5} e^{-r/\xi}
\]

But again, our interest is in large \( r \), where we find, in a reasonably general way:

\[
\langle S(r) S(0) \rangle \propto e^{-r/\xi}
\]

These results were first obtained by van der Waals, and later by Ornstein and Zernike, and Landau. Note, near \( T_c \)

\[
\xi \to \infty \text{ due to } qK_T \to \infty .
\]