1. On the distant planet Tralfamidore, inhabitants are increasingly worried about the possibility of solving the one-dimensional spin-1 Ising model. This model has $N$ spins $S_i$ at every lattice site $i = 1, 2, \ldots N$. The spins can each take the value $S_i = -1, 0, 1$, and interact via the energy

$$E = -J \sum_i S_i S_{i+1} - H \sum_i S_i,$$  \hspace{1cm} (1)

where $J$ is a positive interaction constant, $H$ is a constant external field, and $S_{N+1} \equiv S_1$.

Show that the partition function can be found from the eigenvalues of a $3 \times 3$ matrix, and find the form of the matrix. You do not need to diagonalize the matrix, just find the form of the matrix.

2. A two-dimensional spin-2 Ising model has $N = L \times L$ spins $S_i$ at every site of a square lattice $i = 1, 2, \ldots N$. The spins can each take the values $-2, -1, 0, 1, 2$, and interact via the energy

$$E = -J \sum_{\langle ij \rangle} S_i S_j - H \sum_i S_i,$$  \hspace{1cm} (2)

where $J$ is a positive interaction constant, $H$ is a constant external field. The interaction is limited to the four nearest neighbors without double counting. The spins on the edges of the square lattice interact via periodic boundary conditions so that, for example, the neighbor to the east of $S_L$ is $S_1$, and the neighbor to the north of $S_L$ is $S_N$, (you do not need this to do this problem, it is mentioned for completeness).

Use mean-field theory to find the average energy per spin and the heat capacity at constant volume.

3. Show that an ideal gas of bosons does not exhibit Bose-Einstein condensation in spatial dimensions $d = 1$ and $d = 2$.

4. Schroeder, problem 7.23 on white dwarf stars (attached).

5. Schroeder, problem 7.24 on neutron stars (attached).
electrostatic forces; the formula actually agrees with experiment, within a factor of 3 or so, for most metals.

**Problem 7.19.** Each atom in a chunk of copper contributes one conduction electron. Look up the density and atomic mass of copper, and calculate the Fermi energy, the Fermi temperature, the degeneracy pressure, and the contribution of the degeneracy pressure to the bulk modulus. Is room temperature sufficiently low to treat this system as a degenerate electron gas?

**Problem 7.20.** At the center of the sun, the temperature is approximately $10^7$ K and the concentration of electrons is approximately $10^{32}$ per cubic meter. Would it be (approximately) valid to treat these electrons as a “classical” ideal gas (using Boltzmann statistics), or as a degenerate Fermi gas (with $T \approx 0$), or neither?

**Problem 7.21.** An atomic nucleus can be crudely modeled as a gas of nucleons with a number density of $0.18$ fm$^{-3}$ (where 1 fm $= 10^{-15}$ m). Because nucleons come in two different types (protons and neutrons), each with spin $1/2$, each spatial wavefunction can hold four nucleons. Calculate the Fermi energy of this system, in MeV. Also calculate the Fermi temperature, and comment on the result.

**Problem 7.22.** Consider a degenerate electron gas in which essentially all of the electrons are highly relativistic ($\epsilon \gg m^2$), so that their energies are $\epsilon = pc$ (where $p$ is the magnitude of the momentum vector).

(a) Modify the derivation given above to show that for a relativistic electron gas at zero temperature, the chemical potential (or Fermi energy) is given by $\mu = \hbar c (3N/8\pi V)^{1/3}$.

(b) Find a formula for the total energy of this system in terms of $N$ and $\mu$.

**Problem 7.23.** A white dwarf star (see Figure 7.12) is essentially a degenerate electron gas, with a bunch of nuclei mixed in to balance the charge and to provide the gravitational attraction that holds the star together. In this problem you will derive a relation between the mass and the radius of a white dwarf star, modeling the star as a uniform-density sphere. White dwarf stars tend to be extremely hot by our standards; nevertheless, it is an excellent approximation in this problem to set $T = 0$.

(a) Use dimensional analysis to argue that the gravitational potential energy of a uniform-density sphere (mass $M$, radius $R$) must equal

$$U_{\text{grav}} = -(\text{constant}) \frac{GM^2}{R},$$

where (constant) is some numerical constant. Be sure to explain the minus sign. The constant turns out to equal 3/5; you can derive it by calculating the (negative) work needed to assemble the sphere, shell by shell, from the inside out.

(b) Assuming that the star contains one proton and one neutron for each electron, and that the electrons are nonrelativistic, show that the total (kinetic) energy of the degenerate electrons equals

$$U_{\text{kinetic}} = (0.0088) \frac{\hbar^2 M^{5/3}}{m_e m_p^{5/3} R^2}.$$
The numerical factor can be expressed exactly in terms of \( \pi \) and cube roots and such, but it's not worth it.

(c) The equilibrium radius of the white dwarf is that which minimizes the total energy \( U_{\text{grav}} + U_{\text{kinetic}} \). Sketch the total energy as a function of \( R \), and find a formula for the equilibrium radius in terms of the mass. As the mass increases, does the radius increase or decrease? Does this make sense?

(d) Evaluate the equilibrium radius for \( M = 2 \times 10^{30} \) kg, the mass of the sun. Also evaluate the density. How does the density compare to that of water?

(e) Calculate the Fermi energy and the Fermi temperature, for the case considered in part (d). Discuss whether the approximation \( T = 0 \) is valid.

(f) Suppose instead that the electrons in the white dwarf star are highly relativistic. Using the result of the previous problem, show that the total kinetic energy of the electrons is now proportional to \( 1/R \) instead of \( 1/R^2 \). Argue that there is no stable equilibrium radius for such a star.

(g) The transition from the nonrelativistic regime to the ultrarelativistic regime occurs approximately where the average kinetic energy of an electron is equal to its rest energy, \( mc^2 \). Is the nonrelativistic approximation valid for a one-solar-mass white dwarf? Above what mass would you expect a white dwarf to become relativistic and hence unstable?

**Problem 7.24.** A star that is too heavy to stabilize as a white dwarf can collapse further to form a **neutron star**: a star made entirely of neutrons, supported against gravitational collapse by degenerate neutron pressure. Repeat the steps of the previous problem for a neutron star, to determine the following: the mass-radius relation; the radius, density, Fermi energy, and Fermi temperature of a one-solar-mass neutron star; and the critical mass above which a neutron star becomes relativistic and hence unstable to further collapse.

**Small Nonzero Temperatures**

One property of a Fermi gas that we **cannot** calculate using the approximation \( T = 0 \) is the heat capacity, since this is a measure of how the energy of the system depends on \( T \). Let us therefore consider what happens when the temperature is very small but nonzero. Before doing any careful calculations, I'll explain what happens qualitatively and try to give some plausibility arguments.

At temperature \( T \), all particles typically acquire a thermal energy of roughly \( kT \). However...