**ADVANCED STATISTICAL MECHANICS PHYS 559:**
**THREE-HOUR FINAL EXAM**

Time: 2:00pm–5:00pm, 12 December 2003
Examiner: Prof. M. Grant
Associate Examiner: Prof. H. Guo

**Instructions:** This exam has four pages. There are six (6) questions. Do any five (5) questions. They all have the same value, although some are easy and some are hard. If you do all six, you get extra marks to a total of no more than 100% on the exam.

There is a two-page formula sheet attached at the end. You may use any of the formulas there without derivation.

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1. If $X$ is an extensive quantity in a system with many independent parts, where $\Delta X$ is the deviation of $X$ from its average value, prove the very handy identity $\langle e^{i\Delta X} \rangle = e^{-\frac{1}{2}\langle(\Delta X)^2\rangle}$.

2. The extra free energy due to a fluctuating surface in the presence of gravity is

$$F = \frac{1}{2} \int d\bar{x} [\sigma (\partial h(\bar{x})/\partial \bar{x})^2 + \rho gh^2],$$

where $\bar{x}$ is the $(d-1)$-dimensional vector in the plane of the interface, $h(\bar{x})$ gives the height of the interface above a reference plane, $L$ is the edge length of the system, $\sigma$ is surface tension, $\rho$ is the constant density difference across the interface, and $g$ is constant gravity.

Calculate the partition function. (Hint: this algebra is very similar to that for working out the renormalized surface tension.)

3. The energy of the spin-1 one-dimensional Ising model is given by

$$E = -J \sum_{i=1}^{N} S_i S_{i+1} - H \sum_{i=1}^{N} S_i,$$

where the $J$ is the interaction constant, $H$ is the constant magnetic field, and the $N$ spins can take the values $-1, 0, +1$, and $S_{N+1} \equiv S_1$.

Show that the partition function can be determined from the largest eigenvalue of a $3 \times 3$ matrix, and work out the form of that matrix.
4. Show that one-dimensional systems cannot have phase coexistence for $T > 0$, if a system has only short-range forces.

Why does this proof break down for systems with infinite range forces (which are, however, largely unphysical)?

5. A silly system is described by

$$F = \int d\vec{x} \left[ \frac{K}{2} (\nabla \psi)^2 + \frac{r}{2} \psi^2 + \frac{u}{6} \psi^6 \right] - H \int d\vec{x} \psi$$

where $r = r_0(T - T_c)$, and $r_0$, $u$, $K$ are positive constants (notice there are six $\psi$’s in the last term), and $H$ is the constant external field. This is the same as what we did in class, except for the $\psi^6$ term.

Use Landau Theory to calculate the critical exponents $\alpha$, $\beta$, $\gamma$, and $\delta$ (these exponents are defined on the formula sheet).

Use Ornstein-Zernicke theory to calculate the correlation function $\langle \psi(x)\psi(0) \rangle$ for $T > T_c$, $T = T_c$, and $T < T_c$.

Are the Landau and the Ornstein-Zernicke theories good approximations in three dimensions (justify your answer)?

6. A renormalization-group transformation $R$ acts to repeatedly coarse-grain a system on a length scale $b > 1$. The coupling constants characterizing a Hamiltonian after $n + 1$ applications of the group are the set $K^{(n+1)}_\alpha$, where the index $\alpha$ runs over all the interaction coupling constants. Hence the transformation can be written

$$K^{(n+1)}_\alpha = R_\alpha(K^{(n)}_\beta),$$

where the fixed point of the transformation is $K^*_\alpha = R_\alpha(K^*_\beta)$.

Linearize around the fixed point with $\delta K^{(n+1)}_\alpha = K^{(n+1)}_\alpha - K^*_\alpha$, and (assuming the matrix describing $R$ is diagonal), obtain the scaling properties and critical exponents near the fixed point in terms of the properties of that matrix. (Hint: this is the same as done in class.)
**SOME HANDY FORMULAE**

The partition function and various probability weights:

\[
Z = \sum_{\text{states}} e^{-E_{\text{state}}/k_B T} = e^{-F/k_B T}.
\]

For a normalized Gaussian, \( \int_{-\infty}^{\infty} dx \, \rho(x) = 1 \) where the weight is

\[
\rho(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}
\]

where \( \langle x \rangle = 0 \), and \( \langle x^2 \rangle = \sigma^2 \). Also:

\[
\rho \propto e^{S/k_B}, \quad \rho \propto e^{-F/k_B T}, \quad \rho = e^{-E_{\text{state}}/k_B T}/Z.
\]

For critical phenomena, the thermodynamic exponents are defined by

\[
C_v \sim |T - T_c|^{-\alpha}, \quad \Delta n \sim |T - T_c|^{\beta}, \quad \kappa_T \sim |T - T_c|^{-\gamma}, \quad \Delta P \sim (\Delta n)^{\delta},
\]

In magnetic systems, \( \kappa_T \to \chi_T, \Delta n \to m, \) and \( \Delta P \to H \). The other two exponents are defined in terms of the correlation function. If \( C(r) = \langle \Delta n(r)\Delta n(0) \rangle \), then

\[
C(r) \sim 1/r^{d-2+\eta}, \text{ for } T = T_c, \quad \text{and } C(r) \sim e^{-r/\xi}, \text{ for } T \neq T_c,
\]

where

\[
\xi \sim |T - T_c|^{-\nu}.
\]

Relations between critical exponents:

\[
\alpha + 2\beta + \gamma = 2, \quad \gamma = \beta(\delta - 1), \quad \gamma = \nu(2 - \eta), \quad \nu d = 2 - \alpha.
\]

Also note the thermodynamic sum rule:

\[
\int d\vec{r} \, C(r) = n^2 k_B T \kappa_T.
\]

Thermodynamic relations:

\[
TdS = dE + PdV, \quad F = E - TS, \quad \kappa_T = -(1/V)(\partial V/\partial P)_T,
\]

\[
P = -(\partial F/\partial V)_T, \quad C_v = -T(\partial^2 F/\partial T^2)_V, \quad (\partial T/\partial V)_S = -(\partial P/\partial S)_V,
\]

\[
(\partial S/\partial V)_T = (\partial P/\partial T)_V, \quad \sigma = (\partial F/\partial A)_{T,V}.
\]
Thermodynamic fluctuations:
\[ \langle (\Delta T)^2 \rangle = k_B T^2 / C_V, \quad \langle \Delta T \Delta V \rangle = 0, \quad \langle \Delta V^2 \rangle = k_B TV \kappa_T. \]

**Fourier relations** (in \( d \) dimensions, change to \((d-1)\) for surface fluctuations, for example). Note all integrals are bounded on small length scales by an ultraviolet cutoff, \(|\vec{k}| < \Lambda\), where \(2\pi/\Lambda \sim (\text{a few nanometers})\), and if necessary on large length scales by an infrared cutoff \(|\vec{k}| > (2\pi/L)\).

\[ \psi(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x}} \hat{\psi}(\vec{k}), \quad \hat{\psi}(\vec{k}) = \int d^d x e^{-i\vec{k} \cdot \vec{x}} \psi(\vec{x}). \]

Closure requires the Dirac delta functions:
\[ \delta(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x}} = \frac{1}{L^d} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{x}}, \quad \delta(\vec{k}) = \int \frac{d^d x}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x}}. \]

The Knoecker delta satisfies:
\[ \delta_{\vec{k},0} = \frac{1}{L^d} \int d^d x e^{i\vec{k} \cdot \vec{x}}, \]
which is 1 if \(\vec{k} = 0\), and zero otherwise. These relations all use the density of states in \(k\) space as \((L/2\pi)^d\). So, the discrete to continuum limit is

\[ \sum_{\vec{k}} \rightarrow \left( \frac{L}{2\pi} \right)^d \int d^d k, \quad \delta_{\vec{k},0} = \left( \frac{2\pi}{L} \right)^d \delta(\vec{k}). \]

and vice versa.

**Handy integrals and sums:**
\[ \int_{-\infty}^{\infty} dx \ e^{-x^2} = \sqrt{\pi}, \quad \sum_{n=0}^{\infty} \frac{y^n}{n!} = e^y, \quad \sum_{n=0}^{\infty} \epsilon^n = 1/(1 - \epsilon), \]
where \(|\epsilon| < 1\).

\[ \exp \left\{ \frac{a}{2N} x^2 \right\} = \int_{-\infty}^{+\infty} dy \sqrt{2\pi / Na} e^{-\frac{N}{2} y^2 + axy} \quad \text{Re } a > 0, \]