1. A flat surface separates two phases of different mass densities \( \rho_l \) and \( \rho_v \), where \( \rho_l > \rho_v \). For example, the two phases could be water and steam. Show that the contribution to the free energy of the surface due to a nonzero gravitational acceleration \( g \) is proportional to \( \int d\vec{x} (\rho_l - \rho_v) g \hbar^2(\vec{x}) \), and find the proportionality constant.

Including this term, and the term we obtained in class involving the surface tension, work out the surface correlation function \( \hat{G}(k) \) and the width \( w \) of the interface.

2. Now, assume you get on a space ship with your water and steam, and go into outer space. Hence, the gravitational field of the earth plays no role. Incorporate the effects of self-gravity, and determine the dependence of the width \( w \) of the interface on the radius \( R \) of the liquid sphere. By self-gravity, I mean the gravitational field set up by the liquid sphere itself, which gives rise to a \( g = g(R) \).

Use the fact (which you should check self-consistently) that \( w/R \ll 1 \): That is, the liquid sphere is so large you can consider it to be locally flat, with an edge length \( L \) determined by \( R \).

Draw the width versus \( R \). What is the maximum value for the width, and at what value of \( R \) does it occur?

3. Unfortunately, your large liquid droplet is contaminated by some oil on the surface, due to a wild party where astronauts were showing off their favorite salad dressing recipes. A circular oil droplet floats on the surface of the water. You can ignore any curvature in the water surface, and just consider the oil droplet. In fact, the droplet’s thickness can be considered negligible, so that it can be treated as a two-dimensional circular droplet of average radius \( \langle R \rangle \) and surface tension (actually line tension) \( \sigma \). The radius fluctuates with angle \( \theta \) in the plane of the interface due to thermal fluctuations, and deviates from its average value by \( \delta R = R(\theta) - \langle R \rangle \). Hence, the width \( w \) is nonzero (again, in the plane of the water’s surface).

Find a way to relate the width to the angular correlations of the radius deviation \( \langle \delta R(\theta)\delta R(\theta') \rangle \). Let \( \delta R(\theta) = \sum_m \delta \hat{R}_m e^{im\theta} \), and calculate the form of \( \langle \delta \hat{R}_m \delta \hat{R}_{m'} \rangle \), and relate the width to this in terms of a sum over \( m \).

Show how the result for \( \hat{G}(k) \) from the lecture notes is recovered from this for \( R \to \infty \). (Note that you will also have to take, and justify, the limit \( m \to \infty \).)