Modelling dielectric heterogeneity in electrophotography

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Abstract
We introduce a continuum model of electrophotographic printing of paper and other disordered composite films. The model allows simulation of any dielectric distribution within a toner transfer gap. It is used to elucidate the role of mass density and paper thickness variations on toner transfer efficiency in Xerographic printing. Our simulations show that spatial variations in inorganic filler distribution within paper as well as paper thickness variations will strongly affect toner transfer uniformity by controlling the point-to-point variation of the local dielectric constant of paper. This variation will manifest itself as a mottle in toned images. It is specifically shown that paper thickness variations in paper lead to toner density variation on the scale of 1 mm, similar to those found in Xerographic printing of commercial papers.

1. Introduction
Recent advances in the science of electrophotography have made Xerographic printing [3, 7, 12] a commercially viable alternative to conventional printing in the area of short-run colour printing. One of the crucial requirements for high quality printing in Xerography lies in the production of uniform solid prints. Since toners penetrate very little into the paper, even after thermal fusing, print density mottle or toner thickness variation is primarily the result of uneven toner transfer onto the paper surface. An important, and poorly understood, cause of uneven toner transfer is that of paper structure heterogeneity, and its effect on the electrostatics of toner transfer. The most common stochastic components of paper structure include mass density, surface roughness, thickness, moisture and filler distribution.

During electrophotographic printing, electrically charged toner particles are transferred to a printing medium (e.g. paper) by an electrostatic field. It is common in designing this process to treat the printing medium as a dielectric of uniform thickness and of a uniform consistency. Inspection of paper, however, reveals a very non-uniform structure [9]. This gives rise to solid Xerographic printed images that contain a significant degree of non-uniformity.
Figure 1. Print density variations in a solid black image printed on commercial paper.

Figure 2. Print density variations in a solid black image printed on a pure cellulose fibre network.

Figure 1 (reprinted from [9]) shows a solid black Xerographically printed square on a portion of commercial paper. The data represents reflected intensity from a white light source, which is mapped onto a grey scale. For clarity of exposition, we present here the data on a black and white scale, with white representing all intensities below a certain threshold and black all the intensities above. The threshold was chosen in the middle range of the original grey scale map. The image shows variations on several length scales, ranging from 100 µm (close to the threshold discernible to the human eye under regular conditions) up to 1 mm. Figure 2 (also reprinted from [9]) shows a printed image under the same condition but onto a laboratory-made hand-sheet, pure cellulose fibre network with isotropic fibre orientations. In contrast to hand-sheets, commercial paper also contain, in addition to cellulose fibres, precipitated calcium carbonate (PCC) filler particles and a high degree of anisotropy in fibre orientations. The figures show that hand-sheet prints are more uniform on longer scales than prints on commercial paper sheets.

This work will investigate how spatial variations in the dielectric properties of paper (or other similar thin-film composites) affect spatial variations in the long-range electrostatic field responsible for toner transfer in Xerography. In particular, we will focus on elucidating the mechanisms for print density variations at the 100 µm to 1 mm scale that have been reported experimentally [9].
2. Models of electrostatic toner transfer

2.1. An effective capacitor model

A Xerographic image (for the case of black and white images) is created when negatively charged toners deposited on a photoconducting surface (drum) are transferred to a paper sheet using a transfer corotron, a wire held at a positive voltage bias that creates an electric field between the photoconducting surface and the wire. This field causes negatively charged toners to detach from the drum and be transferred to the paper substrate. This toned image is then passed through heated fuser rollers which melt it onto the paper substrate. For a uniform paper substrate the simplest, and most common, design model for this process [12] is an effective capacitor model of the toner layer, paper substrate and the bias wire as shown in figure 3. In the figure, the voltage $V$ represents the bias voltage that is applied between the paper and the photoconducting drum. The height $h_p$ represents the thickness of the paper, $h_g$ is the height of the entire paper–air–toner transfer region, $h_t$ is the height of the bottom of the toner layer, which is of thickness $h_g - h_t$. In this work, we further introduce a height $h_c$, within the paper itself. This represents a height that separates a purely cellulose/pre-section of the paper from one that is a mixture of fibre, pores and PCC filler. The inclusion of the height $h_c$ within the paper is used to model the effect of surface-biased filler deposition, a situation common in commercial paper. Electrostatically, the only difference between these regions is their dielectric constant. The dielectric constant of pure cellulose fibre regions is denoted $\varepsilon_c$, while that of the fibre + filler portion is denoted as $\varepsilon_{pf}$. Finally, the dielectric of the toner layer is denoted by $\varepsilon_t$. We denote by $\rho$ the constant coarse-grained charge density of a toner layer and $q_t$, the corresponding charge of an individual spherical toner.

Typical toner sizes are of the order of $10 \, \mu m$ and with a charge of $1.0 \times 10^{-14} \, C$ [11]. As the focus of the work is to examine the role of paper structure, we will hereafter consider toner as a layer $10 \, \mu m$ in thickness with a charge density of $\rho = 1 \times 10^{-6} \, C \, m^{-2}$ (in two dimensions). For this size and charge of toners, the typical force required to detach a $10 \, \mu m$ toner from the drum is roughly $250 \, nN$ [11].

The voltage applied across the transfer gap of figure 3 is equal to the electric field integrated across the transfer gap from $z = 0$ to $h_g$. The potential is thus split among the various layers of the system according to

$$V = \left[ \frac{\sigma}{\varepsilon_c} + \frac{\rho(h_g - h_t)}{\varepsilon_c} \right] h_c + \left[ \frac{\sigma}{\varepsilon_{pf}} + \frac{\rho(h_g - h_t)}{\varepsilon_{pf}} \right] (h_p - h_c) + \left[ \frac{\sigma}{\varepsilon_0} + \frac{\rho(h_g - h_t)}{\varepsilon_0} \right] (h_t - h_p) + \int_{h_t}^{h_g} \left[ \frac{\sigma}{\varepsilon_t} + \frac{\rho}{\varepsilon_t} (z - h_t) \right] dz,$$

(1)
where $\sigma$ is the induced surface charge density on the top plate of the effective capacitor system. The quantities in the brackets $[\cdots]$ correspond to the electric field in the cellulose, PCC-filled paper, air gap and toner layer, respectively. Since the electric field in the toner layer at position $z$ is given by Gauss’s law [8],

$$E = \frac{\sigma}{\varepsilon_1} + \frac{\rho}{\varepsilon_1} (h_g - z),$$

we solve equation (1) for the charge $\sigma$ on the top surface, giving rise to a simple expression for the force per unit charge at position $z$ within the toner layer. This is given explicitly as

$$F_z = -q_t \left\{ \left(V - \rho (h_g - z) \varepsilon_0 \varepsilon_t \right) \left[ \frac{h_c}{\varepsilon_c} + \frac{(h_p - h_c)}{\varepsilon_{pf}} + \frac{(h_g - h_c)}{\varepsilon_1} \right] \left( \frac{(h_g - h_t) h_c}{\varepsilon_c} + \frac{(h_g - h_t) (h_p - h_c)}{\varepsilon_{pf}} + \frac{(h_g - h_t)^2}{2\varepsilon_1} \right) - q_t \left( \frac{\rho (h_g - z)}{\varepsilon_0 \varepsilon_1} \right) \right\}.$$  

Equation (3) is a mean field expression, strictly speaking applicable when the paper substrate is uniform compared with the other elements of the system, such as the toner layer and the photoconductor surface (represented here by the top parallel capacitor plate). In real systems the toner layer and photoconductors vary on length scales below the micrometre range [3], while paper itself can fluctuate both in its mass density and thickness on the scale ranging from $100 \mu m$ (resolution of human reading vision) to tens of millimetres.

It is instructive to consider equation (3) qualitatively. It suggests which paper structures can effect toner transfer. First, there can be significant effects due to variations in paper mass density, which changes its effective dielectric constant. For instance, it has been shown experimentally [13] that PCC filler ($\approx 3$ times a higher dielectric constant than a pure cellulose fibre network) increases the effective dielectric of paper as a function of filler concentration. Furthermore, equation (3) shows that the thickness of paper, $h_p$, plays a critical role. A thicker paper clearly increases the toner transfer strength (i.e. higher electric field in the toner layer) while a thinner paper will reduce toner transfer strength. Most importantly, point-to-point variations in both these quantities are expected to control the spatial variations of toner transfer and print quality in Xerography.

2.2. A continuum model of electrostatic transfer

In this subsection, we introduce a continuum model of electrostatic transfer. We assume that the toner transfer process occurs on timescales sufficiently long that a quasi-steady state electrostatic field is established in the transfer gap. This is a reasonable assumption as the toner transfer process occurs on the order of milliseconds. We begin with the usual equation of electrostatics (Poisson’s equation) within a material $\varepsilon \nabla^2 \phi = -\rho(\vec{x})$ where $\rho(\vec{x})$ is the free charge density in the region. In the case of electrophotography this corresponds to the toner charge distribution. We assume that there are no other free charges within the transfer gap. The constant $\varepsilon$ represents the local dielectric constant, while the function $\phi(\vec{x})$ is the electrostatic potential at any point in space.

If the edges between different domains representing paper, air and toner were sharp, one could, in principle, solve Poisson’s equation in each material and match the solutions according to the well-known boundary conditions of electrostatics. Such a sharp-interface approach is extremely difficult to solve numerically, particularly for two- and three-dimensional disordered and porous systems, involving complex front-tracking which would limit our investigation to unrealistically small system sizes due to the stochastic nature of paper structure.
To eliminate the problem of front tracking, we introduce a continuous dielectric function \( \varepsilon(\vec{x}) \) that continuously interpolates between any two states (materials) within the transfer gap (fibre, filler and air). With this definition of the dielectric, the solution of the electrostatic potential everywhere in the transfer gap is given by

\[
\nabla \cdot (\varepsilon(\vec{x}) \nabla \phi) = -\rho(\vec{x}).
\]

This approach allows for very efficient numerical simulation of the electric field within the transfer gap, without the need for tracking of material boundaries and manual implementation of boundary conditions. It is analogous to the use of thin-interface or boundary-layer models in the study of free surface motion and solidification \([1,5,10]\). One requirement of this method is that the interface width between adjoining materials is sufficiently smaller than the fibre and toner dimensions. When this is the case, it can be rigorously shown \([6]\) that the model in equation (4) satisfies all the usual boundary conditions of electrostatics.

### 2.2.1. Finite element analysis of the continuum model

We simulated equation (4) using a simple relaxation technique applied to a uniform finite element mesh. In this case, the convergence of the electrostatic field evolves as a pseudo-diffusion equation

\[
\frac{\partial \phi}{\partial t} = \nabla (\varepsilon(\vec{x}) \nabla \phi) + \rho(\vec{x}),
\]

where \( t \) is a fictitious time. Describing equation (5) on a finite element mesh of linear isoparametric quadrilateral elements \([4]\), and using the Galerkin weighted residual method gives the following system of numerical matrix equations,

\[
\phi_{n+1} = \phi_n + \Delta t [C^{-1}]^T (-[K] \phi_n^T - [R] + [B.C.]^T)T,
\]

where \([K]\), \([C]\) are analogous to the usual stiffness and mass matrices in diffusion problems, while \([B.C.]\) and \([R]\) are the boundary condition and source matrices. To avoid matrix inversion and large storage problems the matrix \([C]\) is diagonalized using consistent mass lumping \([4]\). Time efficiency of the code was made possible using a new C++ MPI-parallel FEM code we developed and executed on a 16-node Beowulf cluster.

### 3. Paper heterogeneity and toner transfer variations

In this section we use the models described above and analytical methods to examine the role of paper structure variation on toner transfer forces in the transfer gap depicted in figure 3.

#### 3.1. Effect of local filler distribution

We first used the continuum model to investigate how local filler concentration and penetration within the bulk of a paper sheet controls local toner transfer force. We simulated the electrostatic field in the system depicted in figure 4. The transfer gap dimensions are 250 \( \mu \text{m} \) wide by 120 \( \mu \text{m} \) high, with a 2.5 \( \mu \text{m} \) resolution. In the figure, toner comprises a layer of 10 \( \mu \text{m} \) in thickness, dielectric constant \( \varepsilon_t = 4 \) and of charge density of \( \sigma = 1 \times 10^{-4} \text{ C} \mu\text{m}^{-2} \). Paper has a thickness of 102.5 \( \mu \text{m} \) and dielectric constant of cellulose \( (\varepsilon_c \approx 3) \), which is its main constituent. A uniformly random distribution of PCC filler is deposited in a ‘patch’ 100 \( \mu \text{m} \) wide and located near the top surface of the cellulose portion of the paper (see figure 4). Filler particles have a dielectric constant \( \varepsilon_f = 10 \). An air gap of 7.5 \( \mu \text{m} \) separates the two layers, and a potential of +1560 V is applied across the system shown (the bottom plate is held at ground). This would result in an electric field of \( 1.3 \times 10^7 \text{ V} \text{m}^{-1} \) in a capacitor filled only with air and.
Figure 4. System used to study the effect of the filler distribution in electrophotography. The filler patch has dimensions $100 \times 25 \, \mu m^2$ and a uniformly random distribution of filler particles, each of dielectric constant $\varepsilon_f = 10$. The density of fillers is $0.0192 \, \text{particles} \, \mu m^{-2}$. Other parameters are found in table 1.

Table 1. Corresponding parameters in equation (3) for the system of figure 4.

<table>
<thead>
<tr>
<th>$V$ (V)</th>
<th>$h_g$ ((\mu m))</th>
<th>$h_t$ ((\mu m))</th>
<th>$h_p$ ((\mu m))</th>
<th>$h_c$ ((\mu m))</th>
<th>$\varepsilon_t$</th>
<th>$\varepsilon_c$</th>
<th>$\varepsilon_f$ (C m(^{-2}))</th>
<th>$\sigma$ (C m(^{-2}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1560</td>
<td>120.0</td>
<td>110.0</td>
<td>102.5</td>
<td>77.5</td>
<td>4</td>
<td>3</td>
<td>10</td>
<td>$1 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Figure 5. Point-to-point variation of the electrostatic force field corresponding to the conditions shown schematically in figure 4. Different curves correspond to the field at several positions within the thickness of the toner layer.

is a typical field strength in electrophotography [11]. These specific system parameters, which are summarized in table 1, are approximate and can vary in real systems.

Figure 5 shows the electric field within the toner layer depicted in figure 4. The different curves correspond to the field at several vertical positions within the toner layer. Spatial variations in the field correspond to variations in the local dielectric caused by the distribution of the fillers within the paper. The average electrostatic force on toners (over the horizontal
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Figure 6. The magnitude of force on a 10 µm toner as a function of filler number concentration for the three different filler patch configurations. The bottom, middle and top curves correspond to the 100 × 25 µm², 100 × 50 µm² and 100 × 102.5 µm² filler patches, respectively.

extent of the filled region) versus the filler concentration is shown in figure 6 (bottom-most data points). Also shown in figure 6 is the average electrostatic force versus concentration for a filler distribution extending into the z-direction of the paper by 50 µm (middle data points) and 100 µm (top data points).

Figure 6 shows an increase of average toner transfer force with both filler concentration and filler extent into the paper bulk. This is due to an increase in the local effective dielectric constant of paper within the filled region. We also found that increasing filler depth of penetration into the paper tends to average out spatial fluctuations in the electrostatic field within the toner layer. This is due to the long-range nature of the electric field.

These results suggest that optimal toner transfer corresponds to a uniform distribution of filler through the bulk of the paper. Conversely, filler deposited near the surface will decrease toner transfer efficiency and increase spatial variations in local toner transfer. The latter effect is only of practical interest when the filler distribution contains correlations on length scales >100 µm (the threshold of normal vision), as has been reported for certain commercial papers, which contain spatial correlations in their surface filler of the order of 1 mm [9].

We also tested a result of [13] that the effective dielectric of filled paper varies linearly with filler concentration by fitting the data of figure 6 to equation (3). The form for the local dielectric of filled paper was fit to $\varepsilon_{fp} = \alpha \langle C \rangle + \beta$, where $\langle C \rangle$ is the average filler concentration through the filled paper region, and where $\alpha$ and $\beta$ are different for the three filler geometries shown in figure 6. The fits are shown by the solid lines in figure 6. We note that in order to obtain these fits, we adjusted the toner charge density to $3.28 \times 10^{-14}$ C m⁻², a value higher than that of our FEM simulations. We suspect that this is due to the small size of the filled patch relative to the overall transfer gap width being simulated.

3.2. Effect of a general dielectric distribution

To understand the effect of more complex distributions of filler and other constituents (i.e. fibres, porosity and moisture), it is necessary to derive the electrostatic field response that arises as a result of a generalized, random dielectric distribution within the paper substrate. Such a random dielectric distribution may be inferred experimentally from surface potential measurements of paper or any other printing medium.
We begin by denoting a random distribution of dielectric by
\[ \varepsilon(\vec{x}) = \varepsilon_u(z) + \delta\varepsilon_0 \xi(\vec{x}_\perp) \theta(h_p - z), \] (7)
where \( \varepsilon_u(z) = \varepsilon_p \), the average dielectric constant of paper for \( z < h_p \) and, \( \varepsilon_u(z) = \varepsilon_t \), the average dielectric constant of toner for \( z > h_p \) (variables refer to figure 3, with \( h_t = h_p \), a common situation in electrophotography). The function \( \xi(\vec{x}_\perp) \) represents a dimensionless in-plane random distribution that describes the deviation of the dielectric from a uniform value defined by \( \varepsilon_u(z) \). Pre-multiplying \( \xi(\vec{x}) \) by \( \delta \ll 1 \) indicates that the deviation from the average value is small. We also neglect paper thickness in this analytic calculation by pre-multiplying by the step function \( \theta(h_p - z) \). Paper thickness effects will be considered in the next section.

We characterize spatial variations in the electrostatic transfer field at a given height \( z \) in the toner layer via the two-point correlation function, a common approach in the examination of spatial effects in condensed matter systems [2]. This correlation function is denoted as
\[ C(\vec{r}) = \frac{\langle (E_z(\vec{x}) - \langle E_z \rangle)(E_z(\vec{x}') - \langle E_z \rangle) \rangle}{E_0^2}, \] (8)
where \( E_z \) is the \( z \)-component of the electric field, and the angled brackets denote averages over all in-plane positions satisfying \( \vec{x}_\perp - \vec{x}'_\perp = \vec{r} \) (\( z \) is assumed to be fixed at \( z = (h_p)^+ \), just above the paper–toner interface). The constant \( E_0 \) is the electric field at \( z = (h_p)^+ \) in the absence of any dielectric variation (i.e. that due to \( \varepsilon_u(z) \) alone). The function \( C(\vec{r}) \) is thus a measure of how correlated two in-plane points in the electric field separated by a distance \( \vec{r} \) are with each other. For a general in-plane dielectric function \( \xi(x, y) \) equation (8) is given by equation (A21), derived in the appendix (in frequency space).

It is instructive to apply the formalism derived in the appendix to the case of a uniformly random dielectric distribution (i.e. the ‘best case’ situation in commercial paper, where overall filler density and formation is uniform from point-to-point in the sheet). In this case, the distribution \( \xi(\vec{x}) \) (dropping the subscript \( \perp \) ) is uniformly random, exhibiting a two-point density–density correlation function satisfying \( \langle \xi(\vec{x})\xi(\vec{x}') \rangle = \delta(\vec{x} - \vec{x}') \) [2]. The explicit form for the correlation function is given by equation (A23) in the appendix. Figure 7 plots \( C(r) = C(|\vec{x} - \vec{x}'|) \) in frequency space as function for three paper thickness values. The decay of \( C(q) \) represents the (inverse) scale of spatial variations of toner (print) density on the paper surface.

The results of figure 7 are interesting in that they show that even in the case of uniform mass distribution within paper there will be spatial variations in toner transfer force on length scales proportional to the thickness of the paper or, more precisely, the thickness of the paper relative to the thickness of the toner layer. Specifically, toner density variations become uniform on longer length scales with increasing paper thickness.

3.3. Effects of paper thickness heterogeneity on toner transfer forces

This subsection uses the continuum model to numerically simulate the electrostatic transfer forces within a transfer gap simulated using a digitized surface electron microscope (SEM) map of a real section of paper substrate, which exhibits non-uniformity in both mass density and profile thickness. The bottom half of figures 8 and 9 show a cross-section of a paper sheet obtained using a SEM. The section is 7460 \( \mu \)m in length and it is shown in its entirety in the bottom of figure 8, where the height has been increased in scale by 7.5 times. Figure 9 shows a section of the paper at 1 : 1 scale. White pixels in the bottom halves of figures 8 and 9 represent filler, grey is fibre and the solid black is air. The top images of figures 8 and 9 represent the simulated magnitude of the \( z \)-component of the electric field when the paper is inserted between
Figure 7. Two-point correlation functions of the electric field in frequency space at z position just inside the toner layer. The bottom, middle and top curves correspond to paper of thickness 105 µm, 95 µm and 75 µm, respectively.

Figure 8. Electric field magnitude (top) and SEM cross-section (bottom) for a 7460 µm filled-paper substrate. Note the height has been increased in scale by 7.5 times.

Figure 9. Electric field magnitude (top) and SEM cross-section (bottom) for the region 3730–4662 µm of the 7460 µm filled-paper strip, shown in figure 8.
two capacitor plates (i.e. a model transfer gap), where the top plate is grounded and the bottom is held at a potential of 1625 V. For the moment we ignore the presence of toner. The grey scale represents the magnitude of the field, where darker shades correspond to the lowest strength and lightest correspond to the highest field strength. Note that due to shielding, the field inside the sheet is negligible compared to the air gaps above and below the sheet.

Using Photoshop and a custom program, the SEM images were processed such that cellulose, filler and air could be identified and then converted into a dielectric map at a 2.5 µm resolution. In our simulations, the dielectric constant of cellulose was set to $\varepsilon_c = 3$ while that of filler was $\varepsilon_f = 10$ and $\varepsilon_0 = 1$ for air. In what follows we label ‘side 1’ as the bottom side of the sheet cross-section and all in-plane positions $x$ are taken from right to left. ‘Side 2’ corresponds to the top side and the positional results are also taken from right to left. The top and bottom plates (not shown) which establish a potential across the gap would be at the top/bottom boundary of the figures.

The digitized data of figures 8 and 9 were analysed, yielding information about thickness and filler density variations, as well as their corresponding two-point correlation functions. (Two-point correlation functions were calculated as in equation (8), with $E_z$ replaced with the measure of interest.) Figure 10 shows a plot of the local thickness of the paper sample versus position. The corresponding two-point thickness correlation function, shown in figure 11, exhibits a correlation length (decay length in the correlation function) of about $\sim 550 \mu m$.

For each side of the paper sample, the surface filler density was calculated and plotted as a function $x$, the position. Surface filler included all filler particles in a thin layer penetrating a small depth into the paper. Filler layers varying from a thickness of 2 to 40 µm were examined for each side of the paper. Figure 12, shows the distribution of filler density for filler for a thickness 20 µm below side 1 of the paper. Side 2 of the paper displays an analogous filler distribution. We note that filler density was measured near the surface since this is what is actually measured experimentally in many practical cases [9]. Figure 13 shows the two-point correlation function for surface filler on side 1 at two different depths. Point-to-point spatial correlations for this sample are clearly uniform beyond 100 µm. This is in contrast to the measurements of filler concentrations reported in [9], which found a strong correlation on the

![Figure 10. Local thickness of the SEM cross-section of the paper sheet.](image-url)
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Figure 11. Two-point correlation function of the point-to-point thickness variations of paper cross-section.

Figure 12. Number of filler particles from a depth of 20 µm to the top of the surface of side 1.

scale of 1 mm in certain commercial papers. The sample shown here is from an unknown commercial source.

The electrostatic field in the air gap below/above the two paper surfaces was simulated using the continuum model discussed above. The magnitude of this field at the position of the bottom plate is shown in figure 14 (side 1). The electric field at the position of the top plates (side 2) was analogous. The two-point correlation function \( C(r) = C(|\vec{x} - \vec{x}'|) \) was calculated at the position of the bottom and top plates of the transfer gap, and plotted in figure 15. The data show strong correlations up to 500 µm.

Simulations were also performed in a transfer gap that contained a toner layer (with uniform dielectric \( \varepsilon_t = 4 \)) inserted under to the plate corresponding to side 1. The results of toner force variations on side 1 of the paper are essentially identical to the case without a toner layer, except that the force variations inside the toner layer were higher overall magnitude just inside the toner layer due to the shielding introduced by the toners.

The filler density from the simulated paper sample analysed in this subsection is uniform on scales above 100 µm. This fortuitous uniformity in the filler allows us to identify the electric
field variations in the air gap as being created mainly as a result of paper thickness variations, consistent with the prediction in the previous section. The main result of this section is that the Xerographic print density variations, for paper of otherwise uniform mass density, are governed predominantly by paper thickness variations, which were found to be correlated on the scale of ≈0.5 mm (a result typical of many commercial paper sheets). In the case when both paper thickness and filler display variations on visible length scales [9], both factors will contribute to large toner transfer force variations. We note that in this work, as in [9], the overall visual formation\(^3\) was measured and found to vary on scales larger than 5–10 mm, typical of flocs of fibres in paper. We thus conclude that a visual formation does not significantly affect toner transfer variation.

\(^3\) Visual formation is a measure of mass coverage and is not a good indicator of mass density variations when constituents of different density are contained within the paper.
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4. Conclusion

We examined the effects of paper structure on electrophotography by introducing a phase-field type continuum model that allows simulation of any general dielectric distribution in the electrophotographic transfer gap. The model was used to elucidate the role of filler density and paper thickness variations on toner transfer efficiency in Xerographic printing.

Our simulations showed that high-dielectric filler particle concentration affects the local strength of the toner transfer field. We also found that spatial variations in filler density lead to corresponding variations in the electrostatic transfer field. In some commercial papers, where the filler can exhibits spatial variations on scales of the order of 1 mm, our results suggest that filler density variations will play a major role in establishing the pattern of print density mottle in Xerographic solid images [9].

We also examined paper thickness variations, finding that these lead to a separate source of significant variations in the electrostatic field that transfers toner to paper. Simulations of real commercial paper subjected to an electrostatic transfer field revealed that paper thickness variations correlated well with electrostatic field variations on length scales of 0.5 mm.

Our results allowed us to separately identify paper thickness and filler density variations as two attributes of print media that are each responsible for producing toner transfer non-uniformity in commercial Xerographic printing. Further work is required to characterize these two effects when they are present in paper structure concurrently. This can be done easily using the continuum FEM model developed in this work combined with SEM data from a wide range of paper samples.

Acknowledgment

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Appendix A. Linear response theory

In order to focus on variations of the dielectric of the paper, we examine Maxwell’s first equation of electrostatic (equation (4)) with the paper thickness constant. Let the total dielectric constant
be denoted by
\[
\varepsilon(x, y, z) = \varepsilon_0(z) + \delta \varepsilon_1(x, y, z),
\]
where \(\delta\) is a small constant quantifying the deviation of the dielectric constant from a function \(\varepsilon_0(z)\) for which we know how to solve equation (4). We note that \(\varepsilon(x, y, z)\) is in units of the permittivity of free space \(\varepsilon_0^0 = 8.85 \times 10^{-12} \text{ F m}^{-1}\). Inspired from perturbation analysis, we can assume that the perturbation of the ‘dielectric field’ away from \(\varepsilon_0(z)\) will bring about a perturbation of the electric potential as an expansion in powers of the small parameter \(\delta\),
\[
\phi = \phi_0 + \delta \phi_1 + \delta^2 \phi_2 + \cdots .
\]
(A2)
The electric field has a similar expansion since \(E = -\nabla \phi\). Substituting the expansion for \(\varepsilon\) (equation (A1)) and \(\phi\) (equation (A2)) into equation (4), we obtain a set of equations whose solutions in principle give all terms in the expansion of \(\phi\). The lowest order terms, \(\phi_0\) and \(\phi_1\) are given by the solution of the following two equations:
\[
\nabla \cdot (\varepsilon_0(z) \nabla \phi_0) = -\rho(x) = \rho_t
\]
(A3)
and
\[
\nabla \cdot (\varepsilon_0(z) \nabla \phi_1) = -\nabla (\varepsilon_1(x) \nabla \phi_0).
\]
(A4)
We define \(\varepsilon_0(z)\) to describe a uniform paper–toner double layer given by:
\[
\varepsilon_0(z) = \varepsilon_p + (\varepsilon_t - \varepsilon_p) \Theta(z - h_p),
\]
where \(\Theta(z)\) is the unit step function, \(\varepsilon_p\) is the dielectric of paper and \(\varepsilon_t\) is the dielectric of toner. The solution of equation (A3) is then
\[
E_0(z) = -\nabla \phi_0 = \begin{cases} 
-\varepsilon_p, & 0 < z < h_p, \\
-\varepsilon_0 a + \rho_t (z - h_p), & h_p < z < h_g,
\end{cases}
\]
where \(a = a(\varepsilon_p, \varepsilon_t, h_p, h_g, V)\) is a constant that depends on toner and paper dielectric constants, the height of the paper layer, the thickness of the transfer gap and the bias voltage, respectively. Its form is not shown here as it will not be used here explicitly.

Equation (A4) is solved using a Green’s function method. Specifically,
\[
\phi_1(x) = -\iint_{\text{Gap}} G(\vec{x}; \vec{x}') \nabla (\varepsilon_1(x) \nabla \phi_0(\vec{x}')) \, dx \, dy \, dz,
\]
where \(G(\vec{x}; \vec{x}')\) is a Green’s function. This is the solution of equation (A4) with the right-hand side replaced by a delta function ‘point source’ \(\delta(\vec{x} - \vec{x}')\), subject to the boundary conditions \(\phi_1 = 0\) at \(z = 0\) and \(z = h_g\). We expand the three-dimensional Green’s function in a two-dimensional Fourier transform
\[
G(\vec{x}; \vec{x}') = \frac{1}{4\pi^2} \int_{k_z} \int_{k_y} g(z; z'; \vec{k}) e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} \, dk_x \, dk_y,
\]
(A8)
where \(\vec{k} \equiv (k_x, k_y)\) is a two-dimensional wave-vector, i.e. \(|\vec{k}| = 2\pi / \lambda\), corresponding to a wavelength \(\lambda\). The function \(g(z; z'; \vec{k})\) is similarly a one-dimensional \(z\)-direction Green’s function corresponding to a given wavevector \(\vec{k}\). Substituting equation (A8) into Green’s function equation, we arrive at a one-dimensional piecewise linear equation for \(g(z; z'; \vec{k})\) since \(\varepsilon_0(z)\) is piecewise constant. The solution requires three additional boundary conditions. These require that: (1) \(g(z; z'; \vec{k})\) be continuous when \(z' = z = h_p\) and (2) Gauss’s law [8] is satisfied at material interfaces, i.e.
\[
\varepsilon_1 \frac{dg}{dz} \bigg|_{z_+} - \varepsilon_2 \frac{dg}{dz} \bigg|_{z_-} = 0,
\]
(A9)
where \( I_p \) represents the top-sided and bottom-sided limit of the matching points \( z' = z \) and \( z' = h_p \). After some lengthy algebra, we arrive at:

\[
g(z; z', \mathbf{k}) = \begin{cases} \frac{\sinh(\mathbf{k}(z - h_p)) \sinh(\mathbf{k}(z - h_p)) \sinh(\mathbf{k}|z'|)}{\Delta}, & 0 < z' < h_p, \\
\frac{\sinh(\mathbf{k}(z - h_p))}{\varepsilon_t \Delta} \left[ \frac{c_1}{|\mathbf{k}|} \sinh(\mathbf{k}(z' - h_p)) + \varepsilon_t \sinh(\mathbf{k}|h_p| \sinh(\mathbf{k}|z' - z|) \right], & h_p < z' < z, \\
c_1 \sinh(\mathbf{k}(z - h_p)) \sinh(\mathbf{k}(z' - h_p)), & z < z' < h_g. \end{cases}
\]

where,

\[
\Delta = c_1 \sinh(\mathbf{k}(z - h_p)) + \varepsilon_t \mathbf{k} \sinh(\mathbf{k}|h_p| \sinh(\mathbf{k}|z - h_g|)),
\]

\[
c_1 = \varepsilon_t |\mathbf{k}| \sinh(\mathbf{k}|h_p|) \cosh(\mathbf{k}(z - h_p)) + \varepsilon_p |\mathbf{k}| \sinh(\mathbf{k}(z - h_p)) \cosh(\mathbf{k}|h_p|).
\]

As an example of this formalism, we consider an in-plane sinusoidal perturbation of the dielectric, which is limited to a thin layer \( h_c < z < h_p \) within the paper. This perturbation is written as

\[
\varepsilon_t(\mathbf{x}) = \frac{1}{4} \varepsilon_0 [\theta(h_p - z) + \theta(z - h_c)] \sin(\mathbf{q} \cdot \mathbf{x}_\perp).
\]

Substituting \( \varepsilon_t \) into equation (A7), we obtain a first-order relation for the electric field in the toner layer,

\[
\varepsilon_t E_z(x, y, z) = \frac{\varepsilon_p}{\varepsilon_t} \left[ 1 + \frac{\delta \varepsilon_t}{4\pi \varepsilon_p} \left( \frac{\partial g(z, h_0^- |\mathbf{q}|)}{\partial z} - \frac{\partial g(z, h_c^- |\mathbf{q}|)}{\partial z} \right) \sin(\mathbf{q} \cdot \mathbf{x}_\perp) \right] E_0. \tag{A14}
\]

where \( \Lambda = 5 \mu m \) (half the size of a toner particle) and where \( E_0 = E_0(z = h_p^-) \) (the notation \( h_0^- \) implies that we use the relevant function on the interval \((0, h_p)\)) is in units of \( \varepsilon_t / \varepsilon_t^0 \) in equation (A14). The expression in the square brackets \([\cdots]\) in equation (A14) can be used to predict the linear response, within the toner layer, for any generalized in-plane dielectric variation within the paper that can be spectrally decomposed.

**Appendix A.1. Derivation of the two-point correlation function**

We next consider the case where \( h_c = 0 \) in equation (A13) and replace sinusoidal perturbation \( \sin(\mathbf{q} \cdot \mathbf{x}_\perp) \) by

\[
\zeta(\mathbf{x}) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{q} e^{-i\mathbf{q} \cdot \mathbf{x}} d\mathbf{q}, \tag{A15}
\]

noting that \( \zeta(\mathbf{x}) \) continues to be pre-multiplied by a small amplitude \( \delta \) and adopting the notation \( \mathbf{x} = (x, y) \) for the remainder of this section. The (random) perturbation created by \( \zeta \) leads to a perturbed three-dimensional field given by \( E \approx -\nabla \phi_0 - \delta \nabla \phi_1 \), as discussed above.

We characterize variations in the \( z \)-component of the electric field (at a height \( z > h_p \)) through the two-point correlation function defined by

\[
C(\mathbf{r}) = \frac{\langle E_z(\mathbf{x}) - \langle E_z \rangle \rangle \langle E_z(\mathbf{r}) - \langle E_z \rangle \rangle}{E_0^2}, \tag{A16}
\]
where the angled brackets denote spatial averages over all in-plane vectors $\vec{x}$ and $\vec{x}'$ such that $\vec{x} - \vec{x}' = \vec{x}_1 - \vec{x}_1' = \vec{r}$. Alternatively, we can consider averages over different configurations of $\zeta$ while $\vec{x}$ and $\vec{x}'$ remain fixed. Substituting the perturbed $E$-field into equation (A16), we obtain, after some straightforward algebra
\[ C(\vec{r}) = \delta_{\vec{r}}^2 \left( \frac{\nabla \phi_1(\vec{x}) \nabla \phi_1(\vec{x}')}{{E_0}^2} \right) \] 
(A17)
where $\nabla z$ denotes the $z$-direction derivative. In the above, we have assumed that the average of the electric field perturbation is zero over the area of the paper cross-section, i.e. $\langle \nabla z \phi_1(\vec{x}) \rangle = 0$. The field $\phi_1$ is given by substituting $\varepsilon_1(\vec{x})$ into equation (A7),
\[ \phi_1 = -\frac{E_0 \varepsilon_0 f}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta_\vec{q} \frac{\partial g(z; h_p, |\vec{q}|)}{\partial z} e^{-i\vec{q} \cdot \vec{x}} d^2 \vec{q}. \] 
(A18)

Substituting for $\nabla z \phi_1(\vec{x})$ into equation (A17) we obtain, after some straightforward algebra, an expression for a structure factor of the electric field (i.e. the Fourier transform of the two-point correlation function),
\[ S(\vec{k}) = \left( \frac{\delta \varepsilon_0 f}{4\pi^2} \right)^2 \langle \zeta(\vec{x}) \zeta(\vec{x}') \rangle^2 \left[ \frac{\partial g(z; h_p, |\vec{k}|)}{\partial z} \right]^2. \] 
(A19)

In obtaining equation (A19) we assume that the correlation function satisfies $C(\vec{x}, \vec{x}') = C(\vec{x} - \vec{x}')$, which implies that electric field variations are homogeneous. In the case of homogeneous dielectric perturbation we can also write
\[ \langle \zeta(\vec{x}) \zeta(\vec{x}') \rangle = Q_{\zeta}(\vec{k} \cdot \vec{x} - \vec{x}'). \] 
(A20)

This allows us to write the structure factor per unit area of the paper as
\[ C(\vec{k}) = \left( \frac{\delta \varepsilon_0 f}{4\pi^2} \right)^2 Q_{\zeta}(\vec{k}) \left[ \frac{\partial g(z; h_p, |\vec{k}|)}{\partial z} \right]^2, \] 
(A21)
where $Q_{\zeta}(\vec{k})$ is the Fourier transform of $\hat{Q}_\zeta(\vec{x})$.

The special case discussed in the text for a uniformly random dielectric distribution corresponds to a dielectric perturbation with correlation function given by
\[ \langle \zeta(\vec{x}) \zeta(\vec{x}') \rangle = \delta(\vec{x} - \vec{x}'), \] 
(A22)
for which $Q_{\zeta}(\vec{k}) = 1$, and
\[ C(\vec{k}) = \left( \frac{\delta \varepsilon_0 f}{4\pi^2} \right)^2 \left[ \frac{\partial g(z; h_p, |\vec{k}|)}{\partial z} \right]^2, \] 
(A23)
which is the function plotted in the text.

References
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