1 Fermionic harmonic oscillator

Free scalar field theories reduce to a product of harmonic oscillators. Fermionic free theories reduce to a product of fermionic harmonic oscillators. These are actually simpler than the SHO, but you may not be familiar with them, so this problem forces you to mess with their properties.

For the fermionic oscillator, there is an operator $b$ together with its Hermitian conjugate $b^\dagger$, which satisfy anticommutation relations,

$$\{b, b\} = 0 = \{b^\dagger, b^\dagger\}, \quad \{b, b^\dagger\} = 1 \quad (1)$$

where as usual $\{c, d\} = cd + dc$. The Hamiltonian is $H = \frac{\hbar}{2}(b^\dagger b - bb^\dagger)$.

Show that there must be a state annihilated by $b$. Hint: what is $b^2$? Name this (properly normalized) state $|0\rangle$.

Argue that $b^\dagger |0\rangle$ cannot vanish. Call this state $|1\rangle$. Show that it is properly normalized.

Now the punchline: show that, provided that all possible operators must be composed of products of $b$ and $b^\dagger$, that $|0\rangle$ and $|1\rangle$ are the only two linearly independent states in the Hilbert space. That is, the operators $b, b^\dagger$ describe a 2-state system. (Hint: build all independent operators; there should only be 4. Show that the two states you found are closed under these operators.)

What is the energy of each state? Does the ground state have positive or negative energy?

1.1 Solution

The Hilbert space should be nonempty so there should be a state $|\psi\rangle$. Either $b|\psi\rangle = 0$, in which case we define $|0\rangle = |\psi\rangle/\sqrt{\langle\psi|\psi\rangle}$, or $b|\psi\rangle = |\chi\rangle \neq 0$, in which case we define $|0\rangle = |\chi\rangle/\sqrt{\langle\chi|\chi\rangle}$. In this latter case, $b|0\rangle \propto b^2|\psi\rangle = 0$, which follows from $2b^2 = \{b, b\} = 0$.

Now we show that $b^\dagger |0\rangle \neq 0$. This follows from

$$bb^\dagger |0\rangle = (1 - b^\dagger b) |0\rangle = |0\rangle - b^\dagger b |0\rangle = |0\rangle - 0 = |0\rangle . \quad (2)$$

Since $|0\rangle$ is a properly normalized state, it is not zero. Name $b^\dagger |0\rangle = |1\rangle$; note that $\langle 1| = (b^\dagger |0\rangle)^\dagger = \langle 0| b$. Therefore

$$\langle 1|1\rangle = \langle 0| bb^\dagger |0\rangle = \langle 0| |0\rangle = 1 , \quad (3)$$

so $|1\rangle$ is properly normalized.
We already found that $b|0\rangle = 0$ and $b|0\rangle = |1\rangle$. Let us find $b|1\rangle$ and $b^\dagger|1\rangle$:

$$b^\dagger|1\rangle = b^\dagger b^\dagger |0\rangle = 0 |0\rangle = 0$$

where we again used $2(b^\dagger)^2 = \{b^\dagger, b^\dagger\} = 0$. Similarly

$$b|1\rangle = bb^\dagger |0\rangle = (1 - b^\dagger b) |0\rangle = |0\rangle - 0 = |0\rangle .$$

So the subspace spanned by $|0\rangle$ and $|1\rangle$ is closed under the action of $b$ and $b^\dagger$. Therefore it is closed under the action of any product of these operators. If arbitrary products of $b$ and $b^\dagger$ are the only possible operators, this pair of states spans the Hilbert space.

An alternative approach is to ask what is the largest possible set of operators one can build from $b$ and $b^\dagger$. The answer is that there are only four operators:

$$1, \ b, \ b^\dagger, \ \text{and} \ b^\dagger b.$$ (6)

The operator $bb^\dagger$ can be expressed as the linear combination $1 - b^\dagger b$. Any product of 3 or more $b, b^\dagger$ can be shortened by using either $b^2 = 0$, $(b^\dagger)^2 = 0$, $b^\dagger bb^\dagger = b^\dagger - b(b^\dagger)^2 = b^\dagger$, or $bb^\dagger b = b - b^\dagger b^2 = b$; so all products of $b, b^\dagger$ can be reduced to sequences of length two or less, leading always to linear combinations of these four operators. The number of independent operators equals the square of the dimension of the Hilbert space, so the Hilbert space must have only 2 independent elements.

The energies are found by acting $H$ on each state:

$$\frac{\omega}{2} \left( b^\dagger b - bb^\dagger \right) |0\rangle = -\frac{\omega}{2} bb^\dagger |0\rangle = -\frac{\omega}{2} b |1\rangle = -\frac{\omega}{2} |0\rangle$$ (7)

showing that $|0\rangle$ has energy $-\omega/2$. Similarly

$$\frac{\omega}{2} \left( b^\dagger b - bb^\dagger \right) |1\rangle = \frac{\omega}{2} b^\dagger |0\rangle = \frac{\omega}{2} |1\rangle$$ (8)

showing that $|1\rangle$ has energy $+\omega/2$. The ground state is $|0\rangle$ and has negative energy, unless $\omega < 0$ in which case $|1\rangle$ is the ground state and it has negative energy.

## 2 Dirac equation and electron $g$ factor

Verify that the Dirac Lagrangian density

$$L[\Psi] = i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi$$ (9)

is invariant under the symmetry $\Psi \rightarrow e^{-i\theta} \Psi$, and $\bar{\Psi} \rightarrow e^{i\theta} \bar{\Psi}$ (which follows from complex conjugation). Find the Nöther current associated with this symmetry. (Careful: you have to sum over fields treating $\bar{\Psi}$ and $\Psi$ as independent!)
Add to the Dirac Lagrangian a term $eA_\mu J^\mu$, with $J^\mu$ the Nöther current for the symmetry you just explored, and $A^\mu$ some spacetime-dependent 4-vector (which may be external or may represent another field). Show that this addition does not change the Nöther current you found. Then show that the Lagrangian can be written as

$$\mathcal{L} = \overline{\Psi} (i\gamma^\mu (\partial_\mu - ieA_\mu) - m) \Psi.$$ (10)

Derive the associated Euler-Lagrange equation, which should be

$$(i\partial^\mu + eA^\mu - m)\Psi = 0.$$ (11)

Multiply by $(-i\partial^\mu - eA^\mu - m)$. REMEMBER that the $\gamma$ matrices do not all commute; show that the equation obeyed by $\psi$ is

$$\left( p^2 + m^2 - eF_{\mu\nu}S^{\mu\nu} \right) \Psi = 0$$ (12)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $p_\mu = i\partial_\mu + eA_\mu$ is the kinetic momentum. Here $S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$ as usual.

Now recall that Lorentz transformations are carried out by $S_{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$, with

$$J_i = \frac{\epsilon_{ijk}}{2} S_{jk}$$ (13)

the angular momentum generator, and that the magnetic field is

$$B_i = \frac{\epsilon_{ijk}}{2} F_{jk}.$$ (14)

Rewrite the extra interaction term for the case of magnetic fields as an angular momentum (spin) dot magnetic interaction.

In the nonrelativistic limit, $m \gg \vec{p}, B$, what is the energy shift to an electron in the presence of a magnetic field? This shift is usually parametrized as

$$-\frac{ge}{2m} \vec{s} \cdot \vec{B}$$ (15)

with $g$ a constant to be determined, and $\vec{s}$ the spin operator, $\vec{\sigma}/2$ for a spin-$\frac{1}{2}$ particle. What value do you find for $g$?

### 2.1 Solution

If $\Psi \to e^{-i\theta} \Psi$ and $\overline{\Psi} \to e^{i\theta} \overline{\Psi}$, then

$$\mathcal{L} \to i e^{i\theta} \overline{\Psi} (\gamma^\mu \partial_\mu - m) e^{-i\theta} \Psi = \overline{\Psi} (\gamma^\mu \partial_\mu - m) \Psi$$ (16)
so it is unchanged. The Nöther current is

\[ j^\mu = -i \frac{\delta L}{\delta \partial^\mu \Psi} \Psi + i \bar{\Psi} \frac{\delta L}{\delta \partial^\mu \bar{\Psi}} = -i(\bar{\Psi} \gamma^\mu \Psi) + 0 = \bar{\Psi} \gamma^\mu \Psi. \] (17)

Adding \( e A^\mu J_\mu \) to the Lagrangian, we get

\[ L = \bar{\Psi} (i \gamma^\mu \partial^\mu - m) \Psi + \bar{\Psi} \gamma^\mu e A^\mu \Psi = \bar{\Psi} (i \gamma^\mu (\partial^\mu - ie A^\mu) - m) \Psi. \] (18)

The Nöther current is not changed because the new term still obeys the symmetry and does not change the \( \partial^\mu \Psi \) or \( \partial^\mu \bar{\Psi} \) dependence of the Lagrangian.

The Euler-Lagrange equation obtained through variation with respect to \( \bar{\Psi} \) is

\[ 0 = \frac{\delta L}{\delta \bar{\Psi}} = (i \gamma^\mu (\partial^\mu - ie A^\mu) - m) \Psi = 0. \] (19)

Multiplying by \( -i \partial^\mu - e \partial^\mu A^\mu - m \) we obtain

\[ 0 = \left( \partial^\mu - ie \partial^\mu \tilde{A} - e A^\mu \partial^\mu - e^2 A^\mu A^\nu + im \partial^\mu + em \tilde{A} - em A - m^2 \right) \Psi. \] (20)

Here I used that \( m \) is constant so \( \partial m = m \partial \). However \( A \) is not a constant so we cannot do that with it! The terms linear in \( m \) cancel. The other terms untangle as follows:

\[ \partial^\mu \bar{\Psi} = \gamma^\mu \gamma^\nu \partial^\mu \partial^\nu = \frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} \bar{\Psi} = -g^{\mu \nu} \partial^\mu \partial^\nu, \] (21)

where we used that derivatives are symmetric under interchange of order to symmetrize the gamma matrices. Similarly \( A A = -A^\mu A^\mu \). But the behavior of the mixed terms is a little different:

\[ A \partial^\mu = \gamma^\mu \gamma^\nu A^\mu \partial^\nu = \frac{1}{2} \left( \{ \gamma^\mu, \gamma^\nu \} + [\gamma^\mu, \gamma^\nu] \right) A^\mu \partial^\nu \]
\[ = \left( -g_{\mu \nu} + \frac{1}{2} [\gamma^\mu, \gamma^\nu] \right) A^\mu \partial^\nu \]
\[ = -A^\mu \partial^\mu - 2i S^\mu_{\nu \lambda} A^\mu \partial^\nu. \] (22)

Similarly

\[ \partial^\mu A \]
\[ = \gamma^\mu \gamma^\nu \partial^\mu A^\nu = \frac{1}{2} \left( \{ \gamma^\mu, \gamma^\nu \} + [\gamma^\mu, \gamma^\nu] \right) \partial^\mu A^\nu \]
\[ = \left( -g_{\mu \nu} + \frac{1}{2} [\gamma^\mu, \gamma^\nu] \right) \partial^\mu A^\nu \]
\[ = -\partial^\mu A^\mu - 2i S^\mu_{\nu \lambda} \partial^\mu A^\nu. \] (23)
Substituting these expressions into Eq. (20), it becomes

\[ 0 = \left( -\partial_{\mu} \partial^{\mu} + ieA_{\mu} \partial^{\mu} + e^2 A_{\mu} A^{\mu} + m^2 \right) \Psi - ie(-2i)S^{\mu \nu}(A_{\mu} \partial_{\nu} + \partial_{\nu} A_{\mu}) \Psi . \]  

(24)

Now write \( \partial_{\mu} A_{\nu} = (\partial_{\mu} A_{\nu}) + A_{\nu} \partial_{\mu} \). Since \( S^{\mu \nu} \) is antisymmetric, the \( A \partial_{\nu} \) terms cancel. We can also use antisymmetry to rewrite

\[ S^{\mu \nu}(\partial_{\mu} A_{\nu}) = \frac{1}{2} S^{\mu \nu}(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) = \frac{1}{2} S^{\mu \nu} F_{\mu \nu} . \]  

(25)

Therefore the last term in Eq. (24) is \(-eS^{\mu \nu} F_{\mu \nu}\). Using the definition of \( p \) in the problem statement, Eq. (24) is therefore

\[ 0 = \left( p^2 + m^2 - eF_{\mu \nu} S^{\mu \nu} \right) \Psi = 0 \]  

(26)

as expected.

Note that

\[ B_i J_i = \frac{\epsilon_{ijk}\epsilon^{ilm}}{4} F_{jk} S_{lm} \]

\[ = \frac{\delta_{ij}\delta_{km} - \delta_{jm}\delta_{ik}}{4} F_{jk} S_{lm} \]

\[ = \frac{1}{4} (F_{jk} S_{jk} - F_{jk} S_{kj}) = \frac{1}{2} F_{jk} S_{jk} \]  

(27)

so \( F_{\mu \nu} S^{\mu \nu} = 2B_i J_i \) plus terms involving \( E \) and \( K \).

The energy is

\[ E^2 = m^2 - 2eB_i J_i \Rightarrow E \approx m - \frac{1}{2m} 2eB_i J_i \]  

(28)

where I used that \( \sqrt{m^2 - \epsilon} \approx m - \epsilon/2m \) if we can expand in \( m \gg \epsilon \). Comparing to the expected energy shift, we see that \( g = 2 \).

### 3 Gamma matrix identities

Using only that the \( \gamma \) matrices are \( 4 \times 4 \) matrices satisfying the Clifford algebra, and using the definition

\[ \phi \equiv \gamma^\mu a_\mu , \]

verify the following:

\[ \gamma^\mu \gamma^\nu = -4 \]  

(31)

\[ \gamma^\mu \gamma^\nu \gamma^\rho = 2 \gamma^\rho \]  

(32)

\[ \gamma^\mu \phi \gamma^\rho = 4 \phi \gamma^\rho \]  

(33)

\[ \gamma^\mu \phi \gamma^\nu \gamma^\rho = 2 \phi \gamma^\nu \gamma^\rho \]  

(34)
Hint: DO NOT multiply any 4×4 matrices to do this problem! Just use repeatedly that $AB = \{ A, B \} - BA$ and recycle each identity as you prove successive ones.

### 3.1 Solution

All of these involve repeated anti-commuting of one $\gamma$ across another, picking up a $-g^{\mu\nu}$.

\[
\kappa\kappa = k_\mu k_\nu \gamma^\mu \gamma^\nu \\
= k_\mu k_\nu \left( \frac{1}{2} [\gamma^\mu, \gamma^\nu] + \frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} \right).
\] (35)

But $k_\mu k_\nu$ is symmetric so the commutator doesn’t contribute, while the anti-commutator gives

\[
\kappa\kappa = k_\mu k_\nu \frac{1}{2} (-2) g^{\mu\nu} 1 = -k^2 1.
\] (36)

Henceforth I will suppress writing the identity matrix.

Continuing;

\[
\kappa\dot{p}\kappa = k_\mu p_\nu k_\alpha \gamma^\mu \gamma^\nu \gamma^\alpha \\
= k_\mu p_\nu k_\alpha \gamma^\mu (-\gamma^\alpha \gamma^\nu - 2g^{\nu\alpha}) \\
= -\kappa \dot{p} - 2p \cdot \kappa \\
= -2p \cdot \kappa + k^2 \dot{p},
\] (37)

where we used the first identity in the last step.

\[
\gamma^\mu \gamma_\mu = g_{\mu\nu} \gamma^\mu \gamma^\nu = \frac{1}{2} g_{\mu\nu} \{ \gamma^\mu, \gamma^\nu \} = -g_{\mu\nu} g^{\mu\nu} = -g^\mu_\mu = -4.
\] (38)

Between the second and third expressions I used that $g_{\mu\nu}$ is symmetric, so I can symmetrize the other half. This result shows that $-\gamma_\mu \gamma^\mu$ just counts the number of spacetime dimensions.

\[
\gamma^\mu \kappa \gamma_\mu = k_\nu \gamma^\mu \gamma^\nu \gamma_\mu = k_\nu (-\gamma_\mu \gamma^\mu - 2g^\mu_\mu \gamma^\mu) = k_\nu (4\gamma^\nu - 2\gamma_\nu) = 2\kappa.
\] (39)

In the next to last step I used the previous result.

\[
\gamma^\mu \dot{p} \kappa \gamma_\mu = p_\nu k_\alpha \gamma^\mu \gamma^\nu \gamma^\alpha \gamma_\mu \\
= p_\nu k_\alpha \left( -\gamma^\mu \gamma^\nu \gamma^\alpha \gamma_\mu - 2g^\alpha_\mu \gamma^\mu \gamma^\nu \right) \\
= p_\nu k_\alpha \left( \gamma^\mu \gamma^\nu \gamma^\alpha \gamma_\mu + 2g^\alpha_\nu \gamma^\mu \gamma^\alpha - 2\gamma^\alpha \gamma^\nu \right) \\
= p_\nu k_\alpha \left( -4\gamma^\nu \gamma^\alpha + 2\gamma^\nu \gamma^\alpha - 2\gamma^\alpha \gamma^\nu \right) \\
= p_\nu k_\alpha \left( -2 \{ \gamma^\nu, \gamma^\alpha \} \right) \\
= 4p_\nu k_\alpha g^{\nu\alpha} = 4p \cdot k.
\] (40)
The next is more complicated but essentially the same. I will not switch out of slash notation this time, and will use

$$\phi \gamma_\mu = a_\nu \gamma^\nu \gamma_\mu = -2a_\nu g^\nu_\mu - a_\nu \gamma^\mu \gamma^\nu = -2a_\mu - \gamma_\mu \phi,$$  \hspace{1cm} (41)

as well as $\phi \bar{\phi} = -2a \cdot b - \bar{\phi} \phi$. I will also use that $\gamma^\mu \ldots a_\mu = \phi \ldots$

$$\gamma^\mu \phi \bar{k} \phi \gamma_\mu = -\gamma^\mu \phi \bar{k} \gamma_\mu \phi - 2\gamma^\mu \phi \bar{k} q_\mu = -4p \cdot k \phi - 2\phi \bar{k} \phi = -4p \cdot k \phi - 2\phi \left(-2p \cdot k - \bar{k} \phi\right) = 2\phi \bar{k} \phi.$$  \hspace{1cm} (42)