Physics 742  Homework 1

1 Lorentz transform warmup

Consider a boost by amount \( b_1 \) in the \( x \)-direction. Take \( b_1 \) to be small and write out \( \Lambda^{\mu \nu} = \exp \omega^{\mu \nu} \) to second order in \( b_1 \).

Now consider a boost by \( b_2 \) in the \( y \)-direction. Again, write out the matrix form of \( \Lambda^{\mu \nu} \) to second order in \( b_2 \).

Find the product \( \Lambda(b_1)\Lambda(b_2) \) and the product \( \Lambda(b_2)\Lambda(b_1) \), to second order in \( b_1 's \). Show that the difference is of order \( b_1 b_2 \), and looks like an \( \omega^{\mu \nu} \) which generates a rotation. What axis is the rotation about?

2 Commutation relations

Show that the commutation relations

\[
\begin{align*}
\left[ J_i, J_j \right] &= i\epsilon_{ijk} J_k, \\
\left[ J_i, K_j \right] &= i\epsilon_{ijk} K_k, \\
\left[ K_i, K_j \right] &= -i\epsilon_{ijk} J_k,
\end{align*}
\]

(1) (2) (3)
together with the definitions

\[
L_i \equiv \frac{J_i + iK_i}{2}, \quad R_i \equiv \frac{J_i - iK_i}{2}
\]
(4)
give rise to the commutation relations

\[
\begin{align*}
\left[ L_i, L_j \right] &= i\epsilon_{ijk} L_k, \\
\left[ R_i, R_j \right] &= i\epsilon_{ijk} R_k, \\
\left[ L_i, R_j \right] &= 0.
\end{align*}
\]
(5) (6) (7)

3 Majorana identities

Using the following relations for \( \gamma \) matrices,

\[
\begin{align*}
\beta \gamma^\dagger_\mu &= -\gamma_\mu \beta, \\
C \gamma^T_\mu &= -\gamma_\mu C,
\end{align*}
\]
(8) (9)
as well as

\[ \bar{\psi}_1 = \psi_1^\dagger \beta, \]
\[ \bar{\psi}_1^T = -C \psi_1, \]
\[ \psi_1^T C = \bar{\psi}_1. \]

prove these useful relations for Majorana spinors \( \psi_1, \psi_2, \)

\[ \bar{\psi}_1 \psi_2 = + \bar{\psi}_2 \psi_1, \]
\[ \bar{\psi}_1 \gamma^5 \psi_2 = + \bar{\psi}_2 \gamma^5 \psi_1, \]
\[ \bar{\psi}_1 \gamma^\mu \psi_2 = - \bar{\psi}_2 \gamma^\mu \psi_1, \]
\[ \bar{\psi}_1 \gamma^\mu \gamma^5 \psi_2 = + \bar{\psi}_2 \gamma^\mu \gamma^5 \psi_1, \]
\[ \bar{\psi}_1 [\gamma^\mu, \gamma^\nu] \psi_2 = - \bar{\psi}_2 [\gamma^\mu, \gamma^\nu] \psi_1. \]

Hint: you can reverse the order of the operators by transposing: \( \bar{\psi}_1 \gamma^\mu \psi_2 = - \psi_2^\dagger \gamma^\mu \bar{\psi}_1^T. \) The \( - \) sign is from the anticommutation of fermionic operators.

Next, show that

\[ (\bar{\psi}_1 \psi_2)^\dagger = + \bar{\psi}_1 \psi_2, \]
\[ (\bar{\psi}_1 \gamma^5 \psi_2)^\dagger = - \bar{\psi}_1 \gamma^5 \psi_2, \]
\[ (\bar{\psi}_1 \gamma^\mu \psi_2)^\dagger = + \bar{\psi}_1 \gamma^\mu \psi_2, \]
\[ (\bar{\psi}_1 \gamma^\mu \gamma^5 \psi_2)^\dagger = - \bar{\psi}_1 \gamma^\mu \gamma^5 \psi_2, \]
\[ (\bar{\psi}_1 [\gamma^\mu, \gamma^\nu] \psi_2)^\dagger = + \bar{\psi}_1 [\gamma^\mu, \gamma^\nu] \psi_2. \]

Hint: Hermitian conjugation reverses the order of operators and daggers them. The matrix \( \beta \) is Hermitian, \( \beta^\dagger = \beta \). You will also need the relations you found in the first half.

Use these to justify the requirements on the coefficients \( A, B, C, D, \) and \( E \) mentioned in the book under Eq. (1.102).

4 Scalars and symmetries

The kinetic term \( \frac{1}{2} \partial_{\mu} \varphi_i \partial^{\mu} \varphi_i \) for \( N \) real scalar fields is invariant under a symmetry \( \varphi_i \rightarrow O_{ij} \varphi_j \), where \( O^T O = 1 \), \( i, j = 1, ..., N \). These form the group of \( N \times N \) real orthogonal matrices \( O(N) \).
1. Write down the most general renormalizable Lagrangian for two real scalar fields, \( \varphi_1 \) and \( \varphi_2 \), subject to the discrete symmetries \( (\varphi_1, \varphi_2) \rightarrow (-\varphi_1, \varphi_2) \) and \( (\varphi_1, \varphi_2) \rightarrow (\varphi_1, -\varphi_2) \).

2. Re-express this Lagrangian in terms of the complex variables
   \[ \psi = \frac{1}{\sqrt{2}} (\varphi_1 + i\varphi_2) \]
   \[ \psi^* = \frac{1}{\sqrt{2}} (\varphi_1 - i\varphi_2) \]

3. The group \( \mathcal{O}(2) \) is equivalent to the group \( U(1) \). If the \( \mathcal{O}(2) \) transformations are written
   \[ \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \rightarrow \mathcal{O}(\theta) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \mathcal{O}(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]
   find the transformation rules for \( \psi \) and \( \psi^* \).

4. What further restrictions are placed on the Lagrangian by requiring that it be \( \mathcal{O}(2) \) invariant (including interaction terms)?
   Write the resulting Lagrangian in terms of both the variables \( (\varphi_1, \varphi_2) \) and \( (\psi, \psi^*) \).

5 Adjoint representation

Any matrices \( T_a \) satisfying the Lie algebra of a group,

\[ [T_a, T_b] = i f_{abc} T_c, \quad (13) \]

generate a representation of the group. This problem shows that the structure functions themselves provide one such set of matrices. Define \( F_{abc} = -i f_{abc} \) and consider the first index \( a \) to be a label and the second and third indices to be a row and column position.

Using SU(2), because it is simpler, write out the three \( 3 \times 3 \) matrices \( \epsilon^1, \epsilon^2, \) and \( \epsilon^3 \). (For SU(2), the structure function \( f_{abc} = \epsilon_{abc} \).) Verify that these matrices in fact satisfy the commutation relations of the Lie algebra, that is, that

\[ [\epsilon^a, \epsilon^b] = i \epsilon_{abc} \epsilon^c. \quad (14) \]

Next, prove the Jacobi identity,

\[ [[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0, \quad (15) \]

which is just a consequence of writing out every term longhand and canceling like terms. What condition does the Jacobi identity imply on the coefficients \( f_{abc} \)?
Finally, show that the antisymmetry of the $f_{abc}$, together with the Jacobi identity, proves that

$$\left[ F^a, F^b \right] = i f_{abc} F^c$$

(16) holds in any group. Therefore the structure functions themselves provide a representation, called the adjoint representation.