Homework set 1: Hamiltonian and initial value formulations of GR

1.) Charges and the Landau-Lifshitz pseudo-tensor

An alternative approach to defining conserved charges in GR comes from viewing GR as a field theory on Minkowski spacetime. In this setting one may define charges in the usual way from the gravitational “stress tensor.” In this problem you will show that these charges are equivalent the ADM charges.

Let \((M, g_{\mu\nu})\) be a 4-dimensional globally hyperbolic spacetime. Decompose the metric \(g_{\mu\nu}(x) = \eta_{\mu\nu}(x) + \gamma_{\mu\nu}(x)\) with \(\eta_{\mu\nu}(x)\) a flat metric. (We make no assumption about the relative strength of \(\eta_{\mu\nu}(x)\) and \(\gamma_{\mu\nu}(x)\).) Consider the expansion of the Einstein tensor in powers of \(\gamma_{\mu\nu}(x)\):

\[
G_{\mu\nu}[g] = \sum_{n=1}^{\infty} G^{(n)}_{\mu\nu}[\gamma], \quad G^{(n)}_{\mu\nu}[\gamma] = O(\gamma^n). \tag{1}
\]

From the Bianchi identity \(\nabla^\mu G_{\mu\nu}(x) = 0\) it follows that each term in this series satisfies \(\nabla^\mu G^{(n)}_{\mu\nu}[\gamma] = 0\), where \(\nabla^\mu\) is the covariant derivative compatible with the full metric \(g_{\mu\nu}(x)\).

a) Derive/obtain/steal the expression for \(G^{(1)}_{\mu\nu}[\gamma]\).

b) Show that \(\eta^{\mu\nu} \nabla_\mu G^{(1)}_{\nu\lambda}[\gamma] = 0\), where \(\nabla_\mu\) is the covariant derivative compatible with the flat metric \(\eta_{\mu\nu}(x)\).

Define the Landau-Lifshitz pseudo-tensor

\[
T^{LL}_{\mu\nu}(x) := -\frac{1}{8\pi G} \sum_{n=2}^{\infty} G^{(n)}_{\mu\nu}[\gamma]. \tag{2}
\]

Then Einstein’s equations are

\[
G^{(1)}_{\mu\nu}(x) = 8\pi G \left( T^{LL}_{\mu\nu}(x) + T^{\text{matter}}_{\mu\nu}(x) \right), \tag{3}
\]

and it follows that when the equations of motion are satisfied

\[
\eta^{\mu\nu} \nabla_\mu \left( T^{LL}_{\nu\lambda}(x) + T^{\text{matter}}_{\nu\lambda}(x) \right) = 0. \tag{4}
\]

Note that \(G^{(1)}_{\mu\nu}[\gamma], T^{LL}_{\mu\nu}(x)\) transform as tensors under Poincaré transformations, but do not transform as tensors under more general coordinate transformations.
We may use Killing vector fields belonging to $\eta_{\mu\nu}$ to construct conserved quantities in the same way we do so in non-gravitational physics, i.e.,

$$Q[\xi] = \int_{\Sigma} n^\mu \xi^\nu \left( T_{\mu\nu}^{\text{L}}(x) + T_{\mu\nu}^{\text{matter}}(x) \right),$$

with $\Sigma$ a Cauchy surface with respect to flat metric and $n^\mu$ the forward-pointing unit normal vector.

Suppose that the spacetime is asymptotically flat so that at large radius $r$ (defined via the parameterization of $\eta_{\mu\nu}(x)$) we have $\gamma_{\mu\nu}(x) = O(r^{-1})$.

c) Show that $Q[\xi]$ reduces to a boundary term on $\partial\Sigma$. It may be useful to consider separately the cases of translations, rotations, and boosts.

d) Moreover, show that for the appropriate $\xi$ the $Q[\xi]$ agree with the ADM charges

$$E_{\text{ADM}} = \lim_{r \to \infty} \int_{S^2} dA \hat{r}^\mu \left( \partial^\nu h_{\mu\nu}(x) - \partial_\mu h_{\nu\nu}(x) \right),$$

$$P_{\mu}^{\text{ADM}} = \lim_{r \to \infty} \int_{S^2} dA \left( K^\mu_{\nu} \hat{r}^\nu - \hat{r}^\mu K \right),$$

where $\hat{r}^\mu$ is the outward-pointing normal of the $S^2$ constant $r$ surface.

2.) **Linearization instabilities in gravity**

Let $(M, g_{\mu\nu})$ be a 4-dimensional spacetime which solves vacuum Einstein equations and admits a $3 + 1$ foliation with compact Cauchy surfaces. We wish to consider perturbations of this spacetime. Let the perturbed spacetime be described in the Hamiltonian formulation by

$$h_{\mu\nu}(x) = h_{\mu\nu}^{(0)}(x) + \kappa h_{\mu\nu}^{(1)}(x), \quad \tilde{\pi}^{\mu\nu}(x) = \tilde{\pi}^{(0)\mu\nu}(x) + \kappa \tilde{\pi}^{(1)\mu\nu}(x), \quad \kappa = \sqrt{8\pi G},$$

where the zeroth-order quantities are those of $g_{\mu\nu}(x)$.

a) Hamilton’s equations at $O(\kappa)$ provide evolution equations for $h_{\mu\nu}^{(1)}(x), \tilde{\pi}^{(1)\mu\nu}(x)$. Convince yourself that these are non-trivial.

b) Let $\xi^\mu(x)$ be a Killing 4-vector field of the background metric $g_{\mu\nu}(x)$; $\xi^\mu(x)$ need not be tangent to the foliations $\Sigma_t$. Show that for such vectors the smeared constraint equations vanish at linear order, i.e.,

$$H[\xi] = \int_{\Sigma} d^3 x \sqrt{h(x)} \left( \xi^\mu \mathcal{H}(x) + \xi_\mu \mathcal{H}(x) \right) = O(\kappa^2).$$

This result shows that when the background solution admits Killing vectors and compact Cauchy surfaces, some of the gravitational constraints are trivial at lowest order in perturbation theory. In such cases, do we trust that the linear perturbations well-approximate non-linear solutions?
3.) The hypersurface deformation algebra

Assume a spacetime admitting a 3 + 1 decomposition with compact Cauchy surfaces $\Sigma_t$. With these boundary conditions, and with $16\pi G = 1$, the gravitational constraints take the smeared forms:

$$S[\phi] = \int_\Sigma d^3x \, \phi(x) \left[ -h^{1/2} R(x) + h^{-1/2} \tilde{\pi}^{\mu\nu}(x) - h^{-1/2} \tilde{\pi}^2(x) \right],$$

$$V[\xi] = -2 \int_\Sigma d^3x \sqrt{h} \, \xi_{\mu} D_{\nu} \tilde{\pi}^{\mu\nu}(x). \quad (9)$$

Here $R(x)$ and $D_{\mu}$ are the Ricci scalar and covariant derivative compatible with $h_{\mu\nu}(x)$, and $\phi(x)$ and $\xi^\mu(x)$ are an arbitrary scalar function and vector field on $\Sigma_t$. The fundamental Poisson brackets are

$$\{ h_{\mu\nu}(x), \tilde{\pi}^{\alpha\beta}(y) \} = \delta_\mu^\alpha \delta_\nu^\beta \delta^{(3)}(x - y), \quad \{ h_{\mu\nu}(x), h_{\alpha\beta}(y) \} = 0, \quad \{ \tilde{\pi}^{\mu\nu}(x), \tilde{\pi}^{\alpha\beta}(y) \} = 0, \quad (10)$$

where $x, y$ are coordinates on $\Sigma_t$ and all tensors are tangent to $\Sigma_t$. Derive the constraint algebra:

$$\{ V[\xi_1], V[\xi_2] \} = V [[\xi_1, \xi_2]],$$

$$\{ V[\xi], S[\phi] \} = S [\mathcal{L}_\xi \phi],$$

$$\{ S[\phi_1], S[\phi_2] \} = V [ h^{\mu\nu} \left( \phi_1 \nabla_\nu \phi_2 \right) ], \quad (11)$$

Hints:

i) When in doubt, integrate by parts.

ii) Note that the conjugate momentum is a tensor density and that the Lie derivative of a tensor density differs from that of a tensor. Using linearity and the Leibnitz property one may derive the Lie derivative of a tensor density from the Lie derivative of more familiar objects.

4.) Initial conditions for black hole and wormhole mergers

In this problem we will construct sets of initial data describing solutions with multiple black holes and wormholes. We will consider asymptotically flat solutions to vacuum Einstein’s equations which are time-symmetric, i.e. they admit a foliation into a family Cauchy surfaces $\{ \Sigma_t \}$ such that $t \to -t$ preserves the foliation (but flips the orientation of $t^\mu$.)

a) Convince yourself that on the $t = 0$ reflection surface $\partial_t h_{\mu\nu} = 0$ and $N_\mu = 0$.

b) Given (a), show that the extrinsic curvature must vanish identically on the reflection surface. Furthermore show that on this surface the momentum constraint is trivially satisfied, and the Hamiltonian constraint reduces to $R[h] = 0$. 

3
The line element on the $t = 0$ surface is $dl^2 = h_{ab}(x)dx^{a}dx^{b}$ with $a = 1, 2, 3$. Consider decomposing $h_{ab}(x)$ into $h_{ab}(x) = \psi^4(x)\delta_{ab}(x)$ with $\delta_{ab}(x)$ a flat Euclidean metric (the choice of the 4th power is unusual in many settings but is convenient here). To enforce asymptotic flatness we require that at large radius $r \to \infty$ we have $\psi(x) = 1 + O(r^{-1})$.

c) Show that the Hamiltonian constraint reduces to the Laplace equation,
\[
\Delta \psi(x) = 0, \tag{12}
\]
where $\Delta$ is the Laplace operator compatible with $\delta_{ab}(x)$.

d) Consider a “point source” solution to the Hamiltonian constraint:
\[
\psi(x) = 1 + \frac{M}{2r}. \tag{13}
\]
This is this solution? What is $M$?

e) Since (12) is a linear, homogeneous differential equation, the sum of any two solutions is another solution. Use this to construct initial data containing multiple black holes. In this case what is the interpretation of the coefficients $M_i$?

f) With the choice (13) the line element diverges at $r = 0$. To see that this is a coordinate artifact, consider changing the radial coordinate to $\hat{r}$ defined by $r = (M/2)^2\hat{r}^{-1}$, and show that this is an isometry of the spatial metric. What is the “shape” of $t = 0$ surface? What is special about the $r = M/2$ surface?

g) The multi-black hole solutions you found in part (e) describe $N$ black holes in one asymptotic region, with each black hole connected via an Einstein-Rosen bridge to another asymptotic region, for a total of $N + 1$ regions. Now construct a solution with two black holes and two asymptotic regions, i.e., a solution for which the two Einstein-Rosen bridges connect to the same asymptotic region. Hints: ii.) the solution you seek has a discrete symmetry which exchanges the two asymptotic regions, ii.) use the method of images.

h) It is also possible to construct multi-black holes solutions with only one asymptotic region. Consider the case of two black holes and further let the parameters $M_1 = M_2$. Use the method of images to construct a solution for which the Einstein-Rosen bridges connect to form a “wormhole.”

i) Finally, why do I describe these sets of initial data as initial conditions for black hole mergers? How do we interpret the portion of the spacetime to the past of the initial data surface?