The Algebraic Approach

Refs: Fewster (1), Wald (2), Hollands+Wald (1), Haag Ch. III. Benini et al.

Where are we?

"Basics" of QFT in CST
- classical field thy
- canonical quantization
- algebraic approach
- UV structure
- "common knowledge" about particles and states

We are a bit behind, so need to reschedule student talks.

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particles, states, KMS
Adam/entanglement
Composite ops./Elisa
BHs
Extra or cancelled
Jerome, Guilherme
Outline:

I. Algebraic approach to QFT in CST
   - motivation
   - the algebra
   - states
   - GNS construction
   - upshot

II. UV structure
   - motivation
   - short-dist. behavior in Minkowski
   - Hadamard condition
   - Ex: α-vacua in dS
I. The Algebraic approach

Motivation

* Have already seen: choice of +/- "frequency" => no unique $H$.

* Fock "vacuum" not meaningful; not unique, and $\Sigma$ can be excited state in another $H$!

* Could also consider (even) quantization on different overlapping charts. How to relate $H$'s?

* Clear there should be some more general structure w/in which all states treated democratically.

* Key observation: algebraic structure satisfied by $\Omega$. $\phi[f]$ the same in all constructions of $H$'s.
The algebra of observables

Recall that from Hamiltonian analysis we obtained set of rules: for \( f \in C^0(M) \)

i.) \( \phi[f] \) is \( C \)-linear (i.e. allow \( f \in C^0(M) \) complex)

ii.) \( \phi[f]^* = \phi[f^*] \)

iii.) \( \phi[0] = 0 \)

iv.) \( [\phi[f_1], \phi[f_2]] = i \mathcal{E}[f_1, f_2] \)

Think of \( \phi[f] \) as symbols for now. Consider finite sum of finite products—of \( \phi[f_i] \). With an appropriate closure (i.e., specification of what \( \in \) sum+products are admissible) these symbols form an abstract algebra. Since we provide a notion of \( \ast \) in (i.), include \( 1 \), it is a "limital \( \Lambda^* \) algebra."

Consider the \( f \in C^0(M)/PC^0(M) \) with \( \text{supp} f \in 0 \). These \( f \) generate a local algebra of observables

\[
\mathcal{A}(0) := \{ \text{free alg}(\phi[f], 1) \mid f \in C^0(0) \}
\]
We will not address choices of closure - but you will have related HW problem.

Local algebras form a "net" which is inductive, i.e. for \( O' \subset O \) then \( A(O') \subset A(O) \). For \( M \), have \( A(M) \) which is union of \( A(O) \ \forall \) open sets \( O \leq M \).

\[
A(M) = \bigcup_{O \leq M} A(O)
\]

Think of \( A(M) \) as list of questions/observed, one may ask in thy. Algebra non-trivial b/c observations can change state, so order matters!

Refer to \( A(O) \), \( A(M) \) as abstract algebras b/c no mention of \( H \) or \( g \), i.e. no representation of \( A(O) \) indicated.

Note: when \( O_1 \cap T^\pm(O_2) \) then all elements of \( A(O_1) \) commute w/ those of \( A(O_2) \).
States

A state is a normalizable, positive, linear functional on algebra $A(M)$.

$$\Psi : A(M) \rightarrow \mathbb{C},$$

$$\langle A^* A \rangle_\Psi \geq 0 \quad \forall A \in A(M),$$

"positivity" constraints

$$\langle \Phi \rangle_\Psi < 1$$

not abbreviated bra-ket notation; this is definition of $\Psi$

In essence, states $\Psi$ just collection of correlation facts.

$$\langle \phi(x) \psi, \langle \phi(x', \phi(x') \rangle_\Psi, .... A \text{ state provides an answer to every question in } A(M).$$

GNS construction

How is this picture related to Hilbert space description like that of canonical quant? It turns out that, given any algebraic state $\Psi$, can construct a $H_\Psi$ on which $\Psi$ is a very "nice" (approximate) state vector.

In essence, we are going to "reverse" process of canonical quant. $\rightarrow$ state first, then $H$. 
**Theorem:** Let $A(M)$ be a unital $*$-algebra and \( \Psi: A(M) \rightarrow \mathbb{C} \) a state. Then $\exists$ a Hilbert space $\mathcal{H}_\Psi$, representation $\pi: A \rightarrow \mathcal{L}(\mathcal{H}_\Psi)$, and a vector $\ket{\Psi} \in \mathcal{H}_\Psi$, such that $\ket{\Psi}$ is cyclic, i.e. $\{\pi(A)\ket{\Psi}\}$ is dense in $\mathcal{H}_\Psi$. GNS triple $\{\mathcal{H}_\Psi, \pi, \ket{\Psi}\}$ is unique up to unitary equivalence.

For proofs see Wald [2], or Benini et al. This is the **GNS construction** of a Hilbert space compatible w/ algebraic state $\Psi$.

Note that canonical quantization is an example of GNS construction.

**Upshot**

- Algebraic approach provides democratic framework for describing all things one might call “states”
- Is a necessary generalization of setting for quantum physics
o **Strengths:**
  - clearly states what questions are
  - omits needless structure
  - manifestly covariant — indeed, no choice of gms!
  - well-adapted to describe sub-regions, subsets of obs.

o **Weaknesses:**
  - formal
  - omits too broad a notion of state!
    - indeed, many pos. lin. functionals on $A(U)$ too singular/strange to be regarded as physical

N.B.:

o what about composite operators? $\phi^2(x)$? $T(n)(x)$?
  Need in order to couple to gravity...

o completion of $*$-algebra not unique. This deeply related to fact that $\phi(x)$ unbounded
  operator. If we switch to "observable" $e^{i\phi(x)}$
  which is bounded can obtain unique $C^*$-
  algebra known as Weyl algebra. See HW.