The initial value problem in GR

Refs: Isenberg, Wald § 10.1-2

- Last time used geometric relations to split EE into constraint + evolution equations.

- Could now attempt to solve EE on computer: feed in initial data, then compute future evolution. But is this problem well-posed? Is solution unique?

- Today we discuss the I.V.P. (Cauchy problem in) GR. In many ways this is more mathematical than we want to get into. Goal is to gain familiarity w/ results, & gain perspective.

Outline:
- Motivation
- Simple example: Newtonian particles
- Review causal structure
- Main theorem for gravity
- Comments & generalizations
Motivation

- as physicists our job is to make quantitive, falsifiable predictions

- initial value problem embodies this task (given $q(t_0), \dot{q}(t_0)$, what is $q(t > t_0)$?)

- any classical theory should admit a "well-posed" IVP.

Well-posedness: We will say the IVP is well-posed if:

i) $\exists$ a sufficiently interesting class of initial data such that the forward evolution of system is uniquely determined.

ii) $\exists$ notion of continuity for map between initial data set $\&$ it's forward evolution. I.e. "small" changes in initial data yield "small" changes to forward evolution

iii) continuity respects causality of relativity (no superluminal propagation)
- Note that for reasons that will become clear, we do not require existence of forward evolution for "all" initial data.

- Do not care about the "health" of future evolution. Presence of singularities in future evolution is generic, not undesirable per se.

- Existence of well-posed IVP not guaranteed (e.g., naive massive tensor field on curved ST not well-posed).

Example: Newtonian particle mechanics $\ddot{\mathbf{F}} = \mathbf{m} \ddot{\mathbf{a}}$

- System of $n$-particles satisfies $n$ $2^{\text{nd}}$ order ODEs:

$$\ddot{\mathbf{q}}_i = F_i(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n)$$

- Use well-known results:

  - for arbitrary $C^1$ data $\mathbf{F}$ unique sol’n over finite interval $\Delta t$ containing initial value surface to (i.)
  - continues in usual topology (ii.)
  - no max. speed in non-relativistic mechanics $\Rightarrow$ no (ii.) requirement
Review of causal structure:

- **Causal curve**: Curve whose tangent vector is everywhere null or timelike

- **Causal future/post \( J^\pm(s) \)**: Set which may be reached from \( S \) by following future/post-directed c.c.
  (also known as domain of influence)

- **Chronological future/post \( I^\pm(s) \)**: replace c.c. w/ timelike-directed curve

- **Achronal set**: no pts. connected by causal curve (also known as "partial Cauchy surface" if closed and has no boundary).

- **Domains of dependence \( D^\pm(S) \)**: given closed achronal set \( S \), \( D^\pm(S) \) is set of all pts \( p \) every inexitable c.c. through \( p \) intersects \( S \).
- Cauchy horizons $H^\pm(s) := \partial D^\pm(s)$

- Cauchy surface: acausal set $\Sigma$ of $M \Rightarrow D(\Sigma) := D^+(\Sigma) \cup D^-(\Sigma) = M$

- a spacetime (region) $(g_{\mu\nu}, M)$ is globally hyperbolic if there exists a Cauchy surface.

**Choquet-Bruhat, Geroch thm**: (Wald thm 10.2.2)

(I will just read this from Wald's text!)

**Comments**:

- There can exist non-globally hyperbolic STS w/ $\Sigma$, $D(\Sigma)$ embeddings. Will see example later w/ charged black holes, when ST may be extended beyond Cauchy horizon of $D(\Sigma)$. Interpretation of these extensions is open...
- This says nothing about “size” of development.

- This guarantees that there exist choices of $N, N^\mu$ that may be used to construct development, but doesn’t guarantee that all choices will succeed. \[ \Rightarrow \exists \text{“bad” choices of gauge} \]

- Corollary: evolution eqs. presence constraints. However, in numerical evolution small errors generated. It turns out that “small” violations of constraints cause “large” problems w/ evolution.

- This creates correspondence between spacetimes $(M, g_{\mu\nu})$ and initial data $(\Sigma, h_{\mu\nu}, K_{\mu
u})$. It can be more useful to work w/ class of initial data, rather than class of spacetimes.

- This may be extended to include matter couplings. Roughly, if $T_{\mu\nu}$ depends only on fields, metric, and 1st derivatives, then $T^\mu_{\nu}$ extends. Examples: Klein-Gordon, Yang-Mills, .... But main tools of this do not work for higher-derivative theories.

- May also generalize to “almost” $C^2$ initial data.