After QFT, where $S[\phi]$ is quadratic in $\phi$, we have no interactions.

Schematically: free theory $H = a^+ a + b^+ b$

$$a^+ |0\rangle \rightarrow a^+ |0\rangle$$

Interacting theory: $H = a^+ a + b^+ b + \lambda (a b^+ + a^+ b^2)$

$$\uparrow$$

Small

$$a^+ |0\rangle \rightarrow e^{i \frac{H t}{\hbar}} (a^+ |0\rangle) \approx a^+ |0\rangle + i \lambda b^+ |0\rangle + \ldots$$

Prob/unit time for $a$-particle $\Rightarrow 2$ $b$ particles.

Slogan: interactions come from cubic & higher order terms in $S[\phi]$.

Lessons:
1. $|0\rangle_{\text{free}} \neq |0\rangle_{\text{interacting}}$.
2. The true vacuum "|0\rangle" is much more complex in interacting QFT.
3. i.e. $a$ & $a^+$ don't really create or destroy particles. This is just an approximation.
A typical interesting theory (scalar)

\[ L = \frac{1}{2} (\partial \phi)^2 - V(\phi) - L_{\text{int}} \]

e.g. \[ L = \frac{1}{2} (\partial \phi)^2 - V(\phi) \] (cf. \[ L = \frac{i}{2} \bar{x}^2 - V(x) \])

\[ \text{If we expand } \quad V(\phi) = V_0 + V_1 \phi + \frac{1}{2} m^2 \phi^2 + \frac{g}{3!} \phi^3 + \frac{\lambda}{4!} \phi^4 + \ldots \]

We can always redefine \( \phi \rightarrow \phi + \text{const} \) to set \( V_1 = 0 \).

We can think of \( V(\phi) \) as potential. Consider \( \phi(t, \mathbf{x}) \) where \( \phi \) is a slowly varying function of \( \mathbf{x} \)

\[ L \approx \frac{1}{2} \dot{\phi}^2 - V(\phi) \]

Energy is minimized \( \Rightarrow \) when \( \phi \) sits at the minimum of \( V(\phi) \)

\( \Rightarrow \) ground state, or "vacuum" where \( V'(\phi_*) = 0 \)

If \( V_1 \neq 0 \), then the field will relax to a valley of \( V(\phi) \) with \( V'(\phi) = 0 \)

\( \Rightarrow \) nearby \( \phi \rightarrow \phi_* \)

\[ V(\phi) \]

\[ "\text{false vacuum.} \quad \text{"true vacuum.} \]

\[ "\text{false vacuum.} \quad \text{"true vacuum.} \]
For now let us just choose field so that the vacuum has $\phi = 0$

Free $\phi$: $\langle 0 | \phi(x) | 0 \rangle = 0$

Interacting QFT, we just shift $\phi$ s.t. $\langle \Sigma | \phi | \Sigma \rangle = 0$

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**Question:** which interaction terms are important? $\phi^3$ or $\phi$

**Dimensional Analysis:**

$$ S = \int d^4x \left( \frac{1}{2} (\partial \phi)^2 - m^2 \phi^2 + \ldots + \sum_i \frac{i}{\lambda_i} \phi^i + \ldots \right) $$

\[
\begin{align*}
[x] & \sim E^{1-1} & [m] & \sim E \\
[\partial x] & \sim E & [\lambda_{i,j}] & \sim E^{4-i-j} \\
[\phi] & \sim E & & \text{i.e. } [\lambda_{i,j}] = 4 - i - j \\
\end{align*}
\]

Imagine an experiment which probes the theory at scale $E$. (e.g. $\phi \phi \rightarrow \phi \phi$ at C.O.M. energy $\mu E$)

$\lambda_{i,j}$ term $\Rightarrow$ correction of order $x \lambda_{i,j} \times \mu^{4-i-j} \Rightarrow \lambda_{i,j} E^{4+i+j} \Rightarrow \lambda_{i,j} E^{-[4+i+j]}$
At low energies, only terms w/ $[\lambda] > 0$ will be important.

**Interactions:**

- $[\lambda] > 0$ relevant
- $[\lambda] < 0$ irrelevant
- $[\lambda] = 0$ marginal

E.g.: to study a scalar theory at low energies, the only relevant interactions are $\phi^3$ & $\phi^4$.

**Idea (Landau):** To study a system at low energies

1. Guess DOF
2. Guess Symmetries
3. Write down the most general action consistent w/ symmetries.
4. Most important contributions from relevant terms.

What sort of symmetries constrain the theory?

E.g.: $\phi \rightarrow -\phi \Rightarrow L = \frac{1}{2}(\partial \phi)^2 - \frac{1}{2}m^2 \phi^2 + \frac{2}{4!} \phi^4 + \ldots$

$\lambda \phi^4$ theory.

$\phi \partial^2 \phi = \text{tot deriv} + (\partial \phi)^2$
The complex scalar $\phi$ with $L_\phi = \frac{1}{2} \partial \phi^\dagger \partial \phi - m^2 |\phi|^2$ is invariant under $\phi \rightarrow e^{i\alpha(x)}\phi$.

$\partial \mu = i(\phi^* \partial_\mu \phi - \partial_\mu \phi^\dagger \phi)$

$Q = \Sigma (a^+ a - b^+ b)$

Idea: We should be able to make different rotations here and in $\alpha$-centauri.

We want $L_\phi$ invariant under $\phi \rightarrow e^{i\alpha(x)}\phi$

$|\phi|^2 = \phi^\dagger \phi$ is ok.

But $\partial_\mu \phi \rightarrow \partial_\mu (e^{i\alpha(x)} \phi) = e^{i\alpha(x)} (\partial_\mu + i\partial_\mu \alpha) \phi$

If we have another field $A_\mu$ s.t. $A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \alpha$

then $D_\mu \phi = (\partial_\mu + ie A_\mu) \phi$ transforms as

$D_\mu \phi \rightarrow e^{i\alpha} D_\mu \phi$

so $(D_\mu \phi)^* D^\mu \phi$ is invariant.

Maxwell theory is invariant under $A_\mu \rightarrow A_\mu - \partial_\mu \Lambda$

Let $\Lambda = \frac{1}{e} \alpha$
The theory:

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_{\mu} \phi^* D^{\mu} \phi - m^2 \phi^* \phi \]

with \( D_{\mu} \phi = (\partial_{\mu} + i e A_{\mu}) \phi \)

is invariant under \( g \to e^i \phi, \quad A_{\mu} \to A_{\mu} - \partial_{\mu} \lambda \)

This is scalar QED.

\[ Q = e \left( 2 a^+ a - b^+ b \right) \] is really the EM charge.

\[ |D_{\mu} \phi|^2 = (\partial_{\mu} - i e A_{\mu}) \phi^* (\partial^{\mu} + i e A^{\mu}) \phi \]

\[ = \partial_{\mu} \partial^{\mu} \phi^* \phi + A_{\mu} e^{i} (\phi^* \partial^{\mu} \phi - \partial^{\mu} \phi^* \phi) \]

\[ + e^2 A_{\mu} A^{\mu} \phi^* \phi \]

\[ L_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^{\mu} A_{\mu} \]

For a Dirac spinor, we can couple the spinor to EM in the same way:

\[ \psi \to e^{i \alpha(x)} \psi \]

\[ D_{\mu} \psi = (\partial_{\mu} - i e A_{\mu}) \psi \]

\[ \text{"covariant derivative"} \]

\[ L_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left( i \gamma^\mu \partial_{\mu} - m \right) \psi \]
\[ \bar{\Psi} (i \gamma^\mu m) \Psi + \Lambda \mu e(\bar{\Psi} \gamma^\mu \Psi) \]

The gauge principle: For any theory with a continuous "global" symmetry parameterized by a continuous parameter \( \Lambda \)
\[ \Phi(x) \rightarrow \Phi(x, \Lambda) \]

Promote to a "local" or "gauge" symmetry by allowing \( \Lambda = \Lambda(x) \) and coupling to a "gauge" field to restore invariance.

\[ \Rightarrow \text{ new force.} \]

All known elementary forces arise in this way.

S.M.: \( U(1) \times SU(2) \times SU(3) \)
\[ \Phi \rightarrow e^{i \lambda} \Phi \]

Promote the Poincaré symmetries to local gauge symmetries:
\[ \phi(x^\mu) \rightarrow \phi(x^\mu + a^\mu(x)) \]
\[ \partial_\mu \phi \rightarrow \nabla_\mu \phi \]

"gauge field" = \( g_{\mu \nu} \)