Feynman Diagrams

The basic observables

\[ \langle 0 | \prod \phi(x_i) \cdots \phi(x_n) | 0 \rangle = \frac{\int \mathcal{D}\phi e^{iS} \phi(x_1) \cdots \phi(x_n)}{\int \mathcal{D}\phi e^{-S}} \]

\[ = \left( \frac{-i}{\pi} \right)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} 2\pi i \int_{J=0} \right] \]

\[ Z[J] = \int \mathcal{D}\phi e^{iS + i\int J(x) \phi(x) dx} \]

In free QFT:

\[ Z[J] = \exp \left( \frac{1}{2} \int d^4x d^4y J(x) \Delta(x-y) J(y) \right) \]

**Free propagator**

\[ Z[J] = \int \mathcal{D}\phi e^{\int \phi (\partial^2 + m^2) \phi + \phi J \sum_j \left( e^{i \int (\partial^2 + m^2)^{-1} J(x)} \right) \]

\[ = \int d^4x J(x) \]
\[ x = \Delta(x-y) \]

\[ Z[S] = \exp \left( \frac{i}{2} \right) \]

**Interacting QFT:**

\[ L = -\frac{1}{2} \phi (\partial^2 + m^2) \phi + L_{\text{int}} \]

For example:

\[ L_{\text{int}} = \frac{1}{3!} g \phi \]

The exact path integral is too hard. But we can take \( g \) is small, and perturb around the free theory...

\[ Z[S] = \int D\phi \exp \left\{ i \int d^4 x \left( L_{\text{free}} + L_{\text{int}} + \frac{1}{2} \phi \right) \right\} \]

\[ = \int D\phi \exp \left\{ i \int d^4 x \left( L_{\text{free}} + L_{\text{int}} \right) \right\} \left( 1 + \frac{ig}{3!} \int d^4 x \phi(x)^3 + \ldots \right) \]

\[ = \exp \left\{ \frac{ig}{3!} \int d^4 x \phi(x)^3 + \ldots \right\} \int D\phi \exp \left\{ i \int d^4 x \left( L_{\text{free}} + L_{\text{int}} \right) \right\} \]

\[ Z[S] = \sum_{N=0}^{\infty} \frac{1}{N!} \left( \frac{ig}{3!} \int d^4 x \phi(x)^3 \right)^N \sum_{p=0}^{N} \frac{1}{p!} \left( \frac{1}{2} \int d^4 y \int d^4 z \Delta(y-z) \right)^{p} \]

If we want to compute an answer, to order \( g^N \), we only include terms \( w/V \in N \).

Similarly, if we are computing an \( n \)-point function, we only need terms \( w/p \leq g^n \).
Each term in \( W \) is represented by a diagram with \( v \) vertices and \( p \) lines.

- Each vertex connects 3 lines, and comes with a factor of \( g \).
- Each line comes with a propagator.

The total number of powers of \( J \) in a term is

\[ E = 2p - 3v. = \text{external sources} \]

\[ Z[J] = \text{sum of diagrams built out of} \]

\[ E = i \int d^4x J(x) \]

\[ V = i g \int d^4x \]

\[ p = \Delta(x-y) \]

\[ \text{e.g.} \]

\[ E = 0: \text{"Vacuum Diagrams" or "Bubble Diagrams"} \]

\[ O(g^2) \]

\[ V = 2 \text{ and } p = 3 \]

\[ \Rightarrow \int d^4x \int d^4y \Delta(x-y)^3 \Rightarrow g^2 \int d^4x \int d^4y \Delta(x-y)^3 \]
\( \Theta(g^4) : V = 4, P = 6 : \)

\[ \text{Diagram} \]

\[ 1E = 2P - 3V \]

\( E = 1 : V = 1, P = 12 : \)

\[ \text{Diagram} \]

“Tadpole”

\( E = 2 : V = 0, P = 1 \)

\[ \text{Diagram} \]

\( V = 2, P = 4 \)

\[ \text{Diagram} \]

“Virtual” Particles

How many terms in expression \( 2 \Sigma \mathcal{J} \) contribute to a given diagram?

1. There are \( 3! \) permutations of \( \frac{\delta}{\delta \mathcal{J}} \) in \((\delta \mathcal{J})^3 \) \( \Rightarrow \) cancel \( \frac{1}{3!} \)
2. " " 2 " " \( \mathcal{J} \) in \( \mathcal{J} A \mathcal{J} \) \( \Rightarrow \) " \( \frac{1}{2} \)
3. There are \( V! \) " of vertices \( \Rightarrow \) cancel \( \frac{1}{V!} \)
Naively, each Feynman diagram has coefficient = 1.

A permutation of a diagram is a rearrangement of vertices, propagators, the 3 lines coming out of a vertex, or the 2 endpoints of a vertex.

\[ \text{symmetries of a diag} = \frac{D!}{S_d} \text{ of permutations that leave a diagram invariant} \]

Rule: ex. Feyn. diag. must be multiplied by \( \frac{1}{S_d} \)

"symmetry factor"

\[ S_d = 2 \times 2 \times 2 \]

\[ S_d = 3! \times 2 \]

\[ S_d = \frac{2^3 \times 2 \times 2}{3!} \]

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