So far we've derived Feynman diagrams for scalar fields. What about more complex theories?

e.g. complex scalar $\phi \rightarrow e^{i\lambda \phi}$

$$L = \partial_{\mu} \phi \partial^{\mu} \phi - m^2 \phi \bar{\phi} + \text{int}$$

$\phi$ = create a particle w/ positive charge or destroys a
anti-particle w/ neg. charge

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2i\hbar} \left( a_p e^{-ipx} + b^*_p e^{ipx} \right)$$

creates a particle
removes an antiparticle.

Can compute correlation functions as in real scalar theory, by taking derivatives of

$$Z[\bar{\phi}, \bar{\varphi}] = \int D\phi e^{iS + \int d^4x \bar{\phi} \mp \bar{\varphi} + \bar{\varphi} \phi - \frac{i}{2} J}$$

The two point function in free QFT

$$\langle T \bar{\phi}(x) \phi(y) \rangle = \int d^4p \ e^{ip(x-y)} \frac{i}{p^2 - m^2 + i\epsilon}$$

$$\langle \phi \phi \rangle = 0$$

$\Rightarrow$ the free propagator is
\[ \bar{\phi}(x) \phi(y) = \langle T \bar{\phi}(x) \phi(y) \rangle \]

"Charge flow" arrow indicates which direction charge flows

"Particle flow" arrow.

Let's assume that \( L_{\text{int}} \) preserves the \( \phi \rightarrow e^+ e^- \phi \) symmetry

\[ \Rightarrow \text{interactions conserve charge} \]

In Feynman rules, we must ensure that diagrams conserve charge

\[ \Rightarrow \text{matching up charge flow arrows}. \]

\[ L_{\text{int}} = \frac{g}{4} (\bar{\phi}(x) \phi(x))^2 = \frac{g}{4} \phi^2 \phi^2 \]

\[ X = \frac{g}{4} \]

\[ \langle \bar{\phi}(x) \phi(x) \rangle = \overrightarrow{\text{propagator}} + \overrightarrow{\text{propagator}} \]

\[ \text{e.g. real scalar } X + \text{ complex } \phi \]

\[ L = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m_x^2 \phi^2 + \bar{\phi} \partial^\mu \phi - m_\phi^2 \bar{\phi} \phi + g \bar{\phi} \phi X \phi \phi \]

\[ \overrightarrow{\text{propagator}} = \phi \phi \text{ propagator} = \frac{i}{p^2 - m_\phi^2 + i\epsilon} \]

\[ \overrightarrow{\text{propagator}} = \phi \phi \text{ propagator} = \frac{i}{p^2 - m_\phi^2 + i\epsilon} \]
Momentum space Feynman rules are as before, but now we have 2 possible incoming particles.

e.g. $q$ creates an $e^+$ or destroys an $e^-$

\[ i\mathcal{M}(e^+ e^- \rightarrow e^+ e^-) = \]

\[ \text{ingoing arrows} \Rightarrow \text{incoming } e^+ \text{, outgoing } e^+ \]

there is no $s$-channel diagram

No "crossing symmetry":

\[ i\mathcal{M}(e^+ e^- \rightarrow e^+ e^-) = \]

\[ \text{incoming electron has charge flow now pointing left} \]

With these rules, $i\mathcal{M}(e^+ e^- \rightarrow e^+ e^-)$ is impossible

i.e.

\[ = 0 \]
These charge flow arrows are not the same as momentum flow arrows.

What about particles w/ spin? e.g. Scalar QED

\[ L = \frac{i}{2} D_{\mu} \phi^{\dagger} (D^{\mu} \phi) - m^2 |\phi|^2 + \frac{i}{2} F^{\mu\nu} F_{\mu\nu} \]

Again this theory can be studied in pert. thy. in the coupling \( e \).

Correlation functions can be computed by taking derivatives of 

\[ Z[J, \bar{J}, J_\mu] = \int D\phi D\bar{\phi} e^{i S + \int d^4x (J_\phi \phi + \bar{J} \bar{\phi} + J_\mu \partial_\mu \phi)} \]

The two point function of \( A_\mu \)

\[ \langle 0 1 | T A_\mu(x) A_\nu(y) | 0 1 \rangle = \int d^4\phi \frac{e^{i p^\mu x - i A_\mu}}{(2\pi)^4} (-i \frac{\epsilon_{\mu \nu}}{p^2 + i\epsilon}) \]

propagator in momentum space, depends on

the choice of gauge...

This form \( a \ g \nu \) is true in "Feynman gauge"...

There are 2 types of interactions:

\[ 1D^2 \phi^2 = \partial \phi \partial \phi - i e A_\mu (\partial_\mu \phi - e_\phi \partial_\mu \bar{\phi}) + \frac{e^2}{4} A_\mu A_\nu \bar{\phi} \phi \]
\[ A_\mu = \text{photon propagator} \]
\[ \not{\mu} \not{\nu} = -i \frac{g_{\mu \nu}}{p^2 + i\varepsilon} \]
\[ \not{\mu} \not{\nu} \not{\rho} = -i e \left( p^\mu + p^\rho \right) \]

in momentum space \( \partial_\mu = i p_\mu \)
\[ \not{\mu} \not{\nu} \not{\rho} = -i e \left( -p^\mu + p^\rho \right) \]

Rule: \(- i e \left( + \text{momentum flows in direction of particle flow} \right) \leftarrow \)
\[ \not{\mu} \not{\nu} \not{\rho} = 2 i e^2 g_{\mu \nu} \text{ "seagull diagram" } \]

Remember that photons have polarizations.
\[ A_\mu = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2w_p} \sum_i \left( \bar{\epsilon}_i^\dagger(p) a_{\mu,i} e^{-ipx} + \epsilon_i(p) a_{\mu,i} e^{+ipx} \right) \]

creates a photon of mom \( \bar{p} \) and polarization \( \epsilon_i \)

Incoming states: \( |p, e\rangle \)
For each incoming state we need a factor of the polarization $\epsilon^\mu$ of the outgoing state.

e.g. amplitude for $8 + e^- \rightarrow e^-$

$$\epsilon_{p \mu} \epsilon^{p+K}_{\mu} = -i e \epsilon^\mu (- (p_\mu) - (p_\mu + k_\mu))$$
$$= i e \epsilon^\mu (2p_\mu + k_\mu)$$

amplitude for $e^- \rightarrow e^- + 8$

$$\epsilon_{p \mu} \epsilon^{p-K}_{\mu} = i e \epsilon^\mu (2p_\mu - k_\mu)$$

We must track indices in Feynman diagrams...

$$\epsilon_i \epsilon_{k_1} \epsilon_{k_4} = \epsilon_i \epsilon_{k_2} \epsilon_{k_3}$$

In the diagram, internal lines get summed over

$$= 2i e^2 g_{\mu \nu} \int d^4k \frac{\epsilon^\mu \epsilon^\nu}{k^2 - i\epsilon}$$