Fermionic Path Integrals

We wish to quantize the Dirac spinor \( \Psi \) using path integrals to derive Feynman rules...

\[
L_{\text{Dirac}} = \overline{\Psi} (i \gamma^\mu D_\mu - m) \Psi \quad \overline{\Psi} = \Psi^\dagger \gamma^0
\]
\[
\gamma^\mu = \gamma_\mu \\delta^\mu
\]

In free theory we quantized by expanding in fermionic oscillators:

\[
\Psi = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} \sum_{s=1}^{2} \left( u_p a_p e^{-i p_x} + v_p b_p e^{i p_x} \right)
\]
\[
\overline{\Psi} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} \sum_{s=1}^{2} \left( -u_p a_p e^{i p_x} + v_p b_p e^{-i p_x} \right)
\]
\[
a_p = \text{create } e^-
\]
\[
b_p^+ = \text{create } e^+ \quad \Theta (x)
\]
\[
\{ a_p , a_p' \} = \delta^{ss'} (2\pi)^3 \delta (p-p') = \{ b_p , b_p' \}
\]

and all others anti-commute...

In the classical limit \( \hbar \to 0 \) the field \( \Psi \) is a classical anti-commuting variable!

Scalar field \( \phi (x) = \) a number per each point in \( \text{ST} \times \)
\( \phi (x) \neq \) a number per point in \( \text{ST} \).
\[ \Psi(x) = \text{a "grassman" number in } \mathbb{ST}. \]

There are two different kinds of numbers:

1) c-numbers (classical number) = commuting numbers.
2) Grassman number.

They have the property that:

1) \[ [c, s] = 0 \]
   c, s are c-numbers
2) \[ [c, \Theta] = 0 \]
   \( \Theta \) is Grassman
3) \[ \varepsilon \Theta, \gamma^2 = 0 \]
   \( \Theta, \gamma \)

We can add & multiply both regular & Grassman numbers:

1) \( c + s \) is c-number
2) \( \Theta + \gamma \) is Grassman
(3) not allowed to add \( c + \Theta \) ) optional
4) \( (c \Theta) \) is Grassman
5) \( (cs) \) is c-number
6) \( \varepsilon (\Theta, \gamma) \) is c-number
Since Grassmanns anti-commute \[ \Theta \Theta^2 = 0 \]
\[ \Rightarrow \Theta^2 = 0 \]

Any Function \[ F(\Theta) = A + B\Theta \]

If \( F \) is c-number \[ A \) c-number, \( B \) Grassmann \]
If \( F \) is Grassmann \[ A \) Grassmann, \( B \) is c-number \]

If \( F \) is c-number, \[ F(\Theta) = A + B\Theta = A - \Theta B \]

\[ \frac{\partial}{\partial \Theta} \]

is defined by

1) \[ \frac{\partial}{\partial \Theta} \Theta = 1 \] \[ \Rightarrow \]

2) \[ \frac{\partial}{\partial \Theta} \]

is Grassmann.

\[ \frac{\partial}{\partial \Theta} (A + B\Theta) = \begin{cases} B & B \text{ c-number} \\ -B & B \text{ Grassmann} \end{cases} \]

\( \Psi(x) = 4 \) Grassmann \( \Psi \)'s per pt in ST.

(c.f. \[ \int dx \times dy = \int dx \times dy \]
\[ dx \times dy = -dy \times dx \]
\[ \vec{v} \times \vec{w} = -\vec{w} \times \vec{v} \) )

We wish to integrate \[ \int dx \Psi \]. So what is \[ \int d\Theta \] ...

\[ \int d\Theta = \text{the analog of } \int_{-a}^{a} dx \]

Require: \( 1) \int d\Theta F(\Theta) = \int d\Theta (A + B\Theta) = \int d\Theta(A) + \int d\Theta(B\Theta) \]

\( 2) \) Invariant under \( \Theta \to \Theta + \gamma \)
i.e. \( \int \delta \infty F(\theta + \eta) = \int \delta \infty F(\theta) \)

(c.f. \( \int \delta \infty f(x) = \int \delta \infty f(x + y) \))

\[
\Rightarrow \int \delta \infty f(\theta + \eta) = \int \delta \infty (A + B \eta + \theta) \\
= \int \delta \infty (A + B \eta) + \int \delta \infty (B \theta) \\
= \int \delta \infty A + \int \delta \infty (B \theta)
\]

\[
\Rightarrow \int \delta \infty A = 0
\]

Def: \( \int \delta \infty (A + B \theta) = B \) "Berezin Integral"

To quantize the free fermion = bunch of gaussian integrals...

\[
\int D\psi D\bar{\psi} \rightarrow \int \delta \infty \bar{\theta} \exp \left\{ -\bar{\theta} B \theta \right\} = \\
= \int \delta \infty \bar{\theta} \left( \frac{\bar{\theta} B \theta}{\bar{\theta} B \theta} \right)^{\frac{1}{2}} = B \uparrow \uparrow \uparrow \\
\bar{\theta} B \bar{\theta} \bar{\theta} \theta = -\bar{\theta} B \theta \theta = 0
\]

\[
c.f. \int \delta \infty e^{-g x^2} = \frac{1}{\sqrt{B}}
\]

\[
\int d\bar{\theta} i \bar{\theta}_j \exp \left\{ -\bar{\theta} B \theta \right\} = \det B
\]

(c.f. \( \int \delta \infty \exp \left\{ -x_i B \frac{\theta_j}{\theta} \right\} \sim \frac{1}{\sqrt{\det B}} \))

More generally: \( \int \delta \infty \delta \bar{\theta}_j \exp \left\{ -\bar{\theta} B \theta + \bar{\theta} i \theta + \frac{\bar{\theta} i \gamma}{\theta} \gamma \right\} \)
\[
= \det B \exp \left( \frac{\bar{\Psi}}{\gamma} B \gamma \Psi \right)
\]

The Dirac spinor can be path integral quantized by taking

\[
\mathcal{Z} [\gamma, \tilde{\gamma}] = \int \mathcal{D} \bar{\Psi} \mathcal{D} \Psi \exp \left\{ i \int d^4 x \bar{\Psi} (i \partial - m) \Psi + \bar{\gamma} \Psi + \bar{\Psi} \gamma \right\} \]

sources which are Grassman spinors

\[
= 2 \mathcal{Z} [0, 0] \exp \left\{ i \int d^4 x d^4 y \tilde{\gamma} (x) S(x, y) \gamma (y) \right\}
\]

where

\[
S(x, y) = \int \frac{d^4 \rho}{(2\pi)^4} e^{i \rho (x-y)} \left( i \frac{\rho + m}{\rho^2 - m^2} \right) \}
\]

is the inverse of \((i \partial - m)\)

\[
(\text{Aside: } \text{ } = \text{ } + \text{ } \cdots)
\]

Just as with scalars, to derive correlation functions take variational derivative w.r.t. \(\frac{\delta}{\delta \bar{\Psi}_{(x)}}\)

\[
\frac{\delta}{\delta \bar{\Psi}_{(x)}} \left( \int d^4 y \left( \bar{\Psi}_y \Psi + \bar{\Psi}_y \gamma \right) \right) = - \bar{\Psi}_{(x)}
\]

and

\[
\frac{\delta}{\delta \bar{\gamma}_{(x)}} \left( \int d^4 y \left( \bar{\gamma}_y \Psi + \bar{\Psi}_y \gamma \right) \right) = \Psi_{(x)}
\]
e.g. \( \langle T \frac{\psi(x)}{a} \frac{\bar{\psi}(y)}{b} \rangle = \frac{1}{i^2} \left( \frac{i \delta}{\delta \bar{\psi}(x)} \right) \left( -i \frac{\delta}{\delta \psi(y)} \right) 2 \left| \begin{array}{c} \delta \theta = \theta = 0 \\
\end{array} \right| \)

\[ = S_{ab}(x-y). \]

General feature: exchanging fermions in a can \( \Rightarrow - \) sign

\( \Rightarrow \) for each fermion in a scattering amplitude, we get a \(-\) sign for exchanging 2 fermions.

\[ \text{e.g.} \quad \begin{array}{c} p_1 \quad \rightarrow \quad \rightarrow \\
\end{array} \quad \begin{array}{c} p_2 \\
\end{array} \]

There will be one more important \(-\) sign for Feynman rules involving fermions \( \Rightarrow \) every loop in a Feynman diagram that involves fermions \( \Rightarrow \) \(-\) sign

\[ \leq \left( \bar{\psi} \psi \right)_1 \left( \bar{\psi} \psi \right)_2 \left( \bar{\psi} \psi \right)_3 \]

\[ \Rightarrow \quad \text{from antiunitary } \psi \]