Yukawa Theory

Before doing full QED, consider re-scaling \( \phi + \text{Dirac fermion} \).

\textit{Yukawa theory:} strong nuclear force. Fermions \( = \text{nucleons} \)

\[ \text{bosons } \phi = \text{pion} \]

\[ \phi = \text{Higgs} \]

\[ L = -\frac{i}{2} \phi (\partial^2 + M^2) \phi + \bar{\Psi} (i \not{\partial} - m) \Psi + \gamma^\mu \bar{\Psi} \gamma^\mu \phi \]

\[ Z [\eta, \bar{\eta}, J] = \int D\phi D\bar{\phi} D\Psi D\bar{\Psi} e^{i S} \] spinors.

\[ = \exp \left\{ i y \int d^4x \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \bar{\phi}(x)} \right\} Z_{\text{free}} \]

\[ \text{where } Z_{\text{free}} = \exp \left\{ i \int d^4x \int d^4y \Delta(x-y) \gamma_\mu \gamma^\mu (x-y) \right\} \]

A correlation function is found by derivatives:

\[ \langle \phi_{(1)} \phi_{(2)} \rangle = \frac{1}{Z} \frac{\delta}{\delta \phi_{(1)}} \frac{\delta}{\delta \phi_{(2)}} \]

\[ J = 0 = \bar{J} = 0 \]

The expansion of the exponentials leads to Feynman graph, just as in the scalar field case.

\[ \text{Posn space: let } \quad \bullet \rightarrow \bullet = \text{scalar source} \]

\[ \bullet \rightarrow \bullet = \text{fermion source} \]

\[ \bullet \rightarrow \bullet = \text{scalar propagator } \Delta \]

\[ \Rightarrow \bullet \rightarrow \bullet = \text{fermion } S_{\alpha \beta} \]
Each interaction + source has a spin labelling the vertices. e.g. $Z = \cdots$ + $\cdots$ + $\cdots$ + $\cdots$. The spin propagators are matrices, and must be multiplied carefully. 

Fermion indices are contracted along lines of charge flow. 

$$< \psi^{(z_1)} \bar{\psi}^{(y_1)} \psi^{(z_2)} \bar{\psi}^{(y_2)} > = \left( ig \right)^2 \int d^4w_1, d^4w_2 \frac{S(x-x_1)}{\delta \eta_x} \frac{S(w_1-w_2)}{\delta \eta_{w_1}} \frac{S(w_2-y_1)}{\delta \eta_{w_2}} \frac{S(y_1-y_2)}{\delta \eta_{y_1}} \Delta_x \Delta_{w_1} \Delta_{w_2} \Delta_{y_1} \Delta_{y_2}$$

$$\bar{\psi} \psi = \left( ig \right)^2 \int d^4w_1, d^4w_2 \text{Tr} \left( \frac{S(w_1-w_2)}{\delta \eta_{w_1}} \frac{S(w_1-w_2)}{\delta \eta_{w_2}} \right) \Delta_{x-w_1} \Delta_{w_2-y_1}$$

Fermion loops $\Rightarrow$ trace over fermion index.
In general, if we permute external states:

- even permutation $\Rightarrow$ no sign
- odd permutation $\Rightarrow$ $-\text{sign}$.

To compute scattering amplitudes, we should use LSZ formalism...

In writing Feynman rules: Notation that incoming particles on L
outgoing particles on R

\[
\Delta(w_1 - w_2)
\]
\[
\rightarrow\quad \text{incoming electron}
\]
\[
\leftarrow\quad \text{position}
\]

Notation: Momentum flows from L $\rightarrow$ R for external legs.

(for legs we will often use notation where momentum flows in
direction of particle flow)

1. Incoming electron $\quad \Psi = \int d^4p \left( U_{s_1} (p_1) + V_{s_1} (p_1) \right)$

2. Outgoing electron

3. Incoming position

4. Outgoing position
7. Join all external lines w/ vertices \( \cdots \quad \gamma = \gamma \) in all ways consistent w/ charge conservation.

8. Each line gets a momentum, with momentum conserved at interactions \( \implies \) tree diagram has no undetermined momenta

Integrate over undetermined momenta in loop diagrams.

9. Internal lines get propagators:

\[
\frac{i}{k^2 - M^2} \quad \rightarrow \quad \frac{i}{\not p - m}
\]

10. Fermion indices contract along lines of charge flow...

Multiply spinor indices starting with a \( \not u \) or a \( \not v \) and following lines if until we get a \( \not u \) or a \( \not v \).

11. \((-1)") for each fermion loop.

12. \((-1)") for each exchange of external fermion legs.

E.g., 2 \( \rightarrow \) 2 scattering of spinors in Yukawa theory...

\[ e^- e^- \rightarrow e^- e^- \quad \downarrow \text{external particle} \]
To make life simple, imagine distinguishable particles

\[ iM = \frac{\epsilon_{a'b'c'd'} \epsilon^{abcs}}{(p_1 - p_1')^2 - M^2} \]

Non-relativistic limit...

\[ p = (m, \overline{p}) + ... \]

\[ \xi^s \text{ is a basis} \]

\[ U_s^s(p) = \frac{1}{m} \left( \begin{array}{c} \xi^s \\ \xi^s \end{array} \right) \]

of 2-component spinors of QM

\[ \hat{U}_s^s = 2m \delta^{ss'} \delta_{nn'} \]

\[ iM = i y^2 \delta^{ss'} \delta_{nn'} \]

\[ \frac{1}{(\vec{p} - \vec{p}')^2 + M^2} \]

since \( (p - p')^2 = -1 \vec{p} - \vec{p}'^2 \)

1) Spin is conserved

2) \( \propto y^2 \) (cf. \( e^2 \) in EM)

In NRQM, scattering of particle from a potential in the Born Approximation is
\[ \langle \vec{p}', \vec{1}; T; \vec{p}, \rangle = -i \mathbf{V} \left( \vec{q} \right) \delta(\vec{E}_p - \vec{E}_{\vec{q}}) \]

Fourier transform of the potential.

\[ \mathbf{V}(\vec{q}) = -\frac{y^2}{\vec{q}^2 + M^2} \]

The Fourier transform is (exercise)

\[ \mathbf{V}(r) = -y^2 \frac{1}{r} e^{-Mr} \]

exponential cutoff.

This is the Yukawa potential.

Nuclear forces fall off rapidly at distances \( \gg 10^{-15} \text{ m} \)

\[ M \approx 200 \text{ MeV} \quad \text{pion} \]

Yukawa is always attractive.