The symmetries of S.R. are the Poisson symmetries, which are generated by:

\[ P^\mu = -i \partial^\mu \]
\[ J_{\mu \nu} = i (x^\mu \partial^\nu - x^\nu \partial^\mu) \]

They act both on classical fields and on the Quantum Hilbert space.

On a scalar field \( \phi(x) \), a finite translation is found by exponentiating \( P \):

\[ T(a^\mu) = e^{i a^\mu P^\mu} : \phi(x) \rightarrow \phi(x + a) \]

\[ e^{i a^\mu P^\mu} \phi(x) = \phi(x + a^\mu \partial^\mu) \phi(x) + \ldots = \phi(x + a) \]

\[ \Lambda = e^{i \omega_{\mu \nu} J^{\mu \nu}} : \phi(x) \rightarrow \phi(\Lambda x) \]

More generally, on some other field \( \Phi(x) \rightarrow M(\Lambda) \Phi(\Lambda x) \)

To find a set of \( M(\Lambda) \), that obey \( M(\Lambda_1, \Lambda_2) = M(\Lambda_1) M(\Lambda_2) \), let's first find a set of \( S^{\mu \nu} \) which obey the same commutation relations as \( J_{\mu \nu} \) and then take

\[ M(\Lambda) = e^{i \omega_{\mu \nu} S^{\mu \nu}} \]

We will find new \( S^{\mu \nu} \).
The group symmetries will act on the Quantum Hilbert Space as unitary operators.

QM: the Hilbert space is a "unitary rep'n" of the symmetry group.

A representation of a group $G$ is a vector space (on a Hilbert Space) $\mathcal{H}$ and a map $M_g : \mathcal{H} \to \mathcal{H}$ s.t. $M_{gh} = M_g M_h$

E.g. Hilbert space of a real scalar field: $|0\rangle$, $|\vec{k}_1\rangle$, $|\vec{k}_1,\vec{k}_2\rangle$...

$T(a) \Rightarrow U(T(a)) |\vec{k}\rangle = \exp\left(ik^\mu a_{\mu}\right) |\vec{k}\rangle$

$\Lambda \Rightarrow U(\Lambda) |\vec{k}\rangle = |\Lambda \cdot \vec{k}\rangle$

2-particle states: $U(\Lambda) |\vec{k}_1, \vec{k}_2\rangle = |\Lambda \cdot \vec{k}_1, \Lambda \cdot \vec{k}_2\rangle$

as a matrix:

$$
\begin{pmatrix}
\vec{k}_1 \\
\vec{k}_2
\end{pmatrix} \mapsto
\begin{pmatrix}
\Lambda & 0 \\
0 & \Lambda
\end{pmatrix}
\begin{pmatrix}
\vec{k}_1 \\
\vec{k}_2
\end{pmatrix}
$$

A rep'n which, by a change of basis, can be made block diagonal is reducible. Otherwise it's irreducible.

Claim: A particle is an irrep. of the Poincare gp.

Scalar: $|\vec{k}\rangle$ 1 DOF per $\vec{k}$

Photon: $|\vec{k},\vec{e}\rangle$ 2 DOF per $\vec{k}$

$\uparrow$ polarization
Theorem (Wigner): The unitary irreps. of $SO(3,1)$ are labelled by 2 pieces of data: $m^2 > 0$, $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

s.t. for fixed $p$, there are $2J + 1$ DoF if $m^2 > 0$

2 DoF if $m^2 = 0$

(Plus other reps apparently not realized in nature)

Photon: $m^2 = 0$ $J = 1$

Graviton: $m^2 = 0$ $J = 2$

What are the rep's of the Lorentz gp?

Idea: instead of using $J^\mu$, use $J^i = \frac{1}{2} \varepsilon^{ijk} J_{jk}$ Rotations $K^i = J^{0i}$ Boosts

Let $N^i = J^i + \imath K^i$

$N^{0i} = J^i - \imath K^i$

The Lorentz algebra becomes: $[N^i, N^j] = \imath \varepsilon^{ijk} N^k$

$[N^i, N^{0i}] = 0$

$[N^{0i}, N^{0i}] = \imath \varepsilon^{ijk} N^k$

Lorentz group $\cong 2$ copies of the rotation group $SU(2)$

$SO(3,1) = SU(2) \times SU(2)_L$ “Left + Right”
A rep' of the Lorentz gp. is given by a pair of reps of SU(2).

\[
\begin{align*}
&\text{SU}(2) : \text{spin}-0, \text{ i.e. "the 1" of SU}(2) \\
&\text{spin}-\frac{1}{2}, \text{ i.e. "the 2" of SU}(2) \\
&\text{spin}-1, \text{ i.e. "the 3" of SU}(2) \\
&\vdots
\end{align*}
\]

A rep' of Lorentz is labeled by a pair \((N_L, N_R)\)

\[
\begin{align*}
(1, 1) &= \text{ scalar} \\
(2, 1) &= \text{ Left handed Weyl Spinor} \\
(1, 2) &= \text{ Right " " " "} \\
(2, 2) &= \text{ 4-vector}
\end{align*}
\]

Dirac Spinor = Left Handed & Right Handed Weyl Spinors.

Field is a 4 component object which we can view as a column vector.
(Dirac) Spinors

A Dirac spinor is a 4-component field $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \Psi_a$.

We will suppress indices whenever possible.

$\Psi$ will transform under Lorentz as $\Psi \rightarrow M(\Lambda) \Psi$

To seek $4 \times 4$ matrices $S^{\mu \nu}$ which obey the Lorentz algebra.

**Trick:** If we have 4 matrices $\delta^\mu$ which obey the "clifford algebra"

\[
\begin{pmatrix} \delta^\mu, \delta^\nu \end{pmatrix} = 2 \langle \delta^\mu, \delta^\nu \rangle 1_{4 \times 4}
\]

then

\[
S^{\mu \nu} = \frac{i}{4} \begin{pmatrix} \delta^\mu, \delta^\nu \end{pmatrix}
\]

Obey the Lorentz algebra.

The gamma matrices will be the following $4 \times 4$ matrices

$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\gamma^i = \begin{pmatrix} 0 & -i \sigma^i \\ i \sigma^i & 0 \end{pmatrix}$

The generator of boosts is

$S^{0i} = \frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

The generator of rotations is

$S^{ij} = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$
Fact: All other $\lambda^M$ that obey the Clifford algebra are related by a change of basis to those above.

The above basis is the "Weyl" basis.

Under a Lorentz transformation, $x^M \rightarrow x'^M = \Lambda^M_N x^N$

$$\Lambda = \exp \left( i \gamma_{\mu \nu} J^{\mu \nu} \right)$$

A Dirac spinor $\Psi(x) \rightarrow \Lambda_{\nu z} \Psi(\Lambda \cdot x)$

where $\Lambda_{\nu z} = \exp \left( i \gamma_{\mu \nu} S^{\mu \nu} \right)$

Goal: Find a good set of EOM obeyed by these $\Psi$

e.g. $(\partial^\mu \partial_\mu + m^2) \Psi = 0$

It's possible to write a 1st order EOM:

$$(i \gamma^\mu \partial_\mu - m) \Psi = 0 \quad " \text{Dirac Eqn}"$$

Check: key fact $\delta^M$ transforms as a vector...

$$\Lambda_{\nu z} \delta^M \Lambda^M_N = \Lambda^M_N \delta^N$$

$\Lambda_{\nu z} = e^{i \gamma_{\mu \nu} J^{\mu \nu}}$

$\Lambda^M_N = e^{i \gamma_{\mu \nu} S^{\mu \nu}}$
\[ \Lambda_{\gamma_2} \gamma^\mu \Lambda_{\gamma_2} = \gamma^\mu + \frac{i}{\hbar} \nabla_\rho \left[ \gamma^\rho, S^\mu \right] + \ldots \]

\[ \Lambda^\mu_\nu \gamma^\nu = \gamma^\mu + \frac{i}{\hbar} \nabla_\rho \left( \gamma^\rho, S^\mu \right) \gamma^\nu + \ldots \]

you will show \[ [\gamma^\mu, S^\nu] = (J^\rho)^{\mu}_{\nu} \gamma^\nu \]

\[ (J^0)_{\nu\rho} = i \left( \eta_{\rho\mu} \delta^\mu_\nu - \delta^\mu_\rho \delta^\nu_\mu \right) \]

So under Lorentz transformation

\[ \gamma^\mu \partial_\mu \Psi \rightarrow \gamma^\mu \left( \Lambda^{-1}_\mu \partial_\nu \right) \Lambda_{\gamma_2} \Psi \]

\[ = \left( \Lambda_{\gamma_2} \Lambda^{-1}_{\gamma_2} \right) \gamma^\mu \Lambda_{\gamma_2} \Lambda_{\gamma_2} \partial_\nu \Psi \]

\[ = \Lambda_{\gamma_2} \gamma^\mu \Lambda_{\gamma_2} \partial_\nu \Psi \]

\[ = \Lambda_{\gamma_2} \gamma^\mu \partial_\nu \Psi \]

If \[ (i \gamma^\mu \partial_\mu - m) \Psi = 0 \] in one frame it is true in all ref. frames.

The Dirac eqn is the Classical EDM of a spin \( \frac{1}{2} \) field of mass \( m \).

Translation: \[ \Psi(x) \rightarrow \Psi(x + a) = e^{i \frac{q_\mu p^\mu}{\hbar}} \Psi(x) \]