Physics 610, Problem Set 1
due: Friday, September 18 at 3:30pm

Please place your completed problem sets in the “Physics 610” box in the physics department mailroom (Rutherford 103b) before the due date. Please do not leave them in my mailbox. You are encouraged to discuss these problems with your colleagues, but you must write up your own solutions; the solutions you hand in should reflect your own work and understanding. Late problem sets will be penalized 10% per day late, unless an extension has been obtained from me or the TA before the due date.

Note: This problem set really just a warm-up. We will review some things which you have seen before and get a chance to practise manipulating classical field equations.

1. Fun with Harmonic Oscillators

Consider a harmonic oscillator with frequency $\omega$. The Hamiltonian is $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$.

Define $a = \sqrt{\frac{\omega}{2}}(x + \frac{i}{\sqrt{2}}p)$.

(a) Show that $[a, a^\dagger] = 1$. Show that $H = \omega(N + \frac{1}{2})$ where $N = a^\dagger a$. Show that $a^\dagger$ and $a$ raise and lower the eigenvalue of $N$ by one. Given a state $|n\rangle$ with $N|n\rangle = n|n\rangle$ compute the norm of $a|n\rangle$. Show that the expectation value of $N$ is positive in any state. Use these last three facts to show that $|n\rangle$ must be an integer. Conclude that the normalized energy eigenstates are of the form $|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$ where $a|0\rangle = 0$.

(b) We define the unit normalized coherent state $|\lambda\rangle$ by $a|\lambda\rangle = \lambda|\lambda\rangle$ and $\langle\lambda|\lambda\rangle = 1$. Using the fact that $p = -i\frac{\partial}{\partial x}$, show that the position space wave function $\langle x|\lambda\rangle$ satisfies a particular first order differential equation in $x$. Solve this equation to compute $\langle x|\lambda\rangle$.

(c) Show that the expectation value of $x$ in this state oscillates in time, with an amplitude and frequency which you should compute.

(d) Compute the expectation value of the energy in a coherent state. How is your answer related to the result you found in part (c)? Compute the variance in the expectation value of the energy.

(e) Consider the family of coherent states described above in the $\hbar \to 0$ limit. How must $\lambda$ scale in this limit in order for the energy of a state to remain constant as we take $\hbar \to 0$? Show that the variance in the Energy vanishes like a power of $\hbar$ (which you should compute) as we take $\hbar \to 0$, provided we scale $\lambda$ so that the energy is kept fixed. This indicates that the coherent states will go over to classical states with fixed energy in the $\hbar \to 0$ limit.

2. Phonons & The Debye Model

Consider a one dimensional crystal, which we think of as a line of $N$ particles which are constrained to lie on a line. Denote by $q_n$, $n = 1 \ldots N$ the displacement of the $n^{th}$ particle
from its equilibrium position. In the limit where each particle only interacts with its nearest neighbour, the Hamiltonian will take the form (at leading order)

\[ H = \sum_n \frac{1}{2m} (p_n^2) + \frac{1}{2} m \omega^2 (q_n - q_{n+1})^2 \]

where \( p_n \) is the momentum conjugate to \( q_n \). In order to make this problem easy to solve we will take the crystal to have periodic boundary conditions by setting \( q_{N+1} = q_1 \); this is a little artificial, but any spurious effects introduced by this should disappear in the limit where \( N \to \infty \).

(a) This problem is easy to solve if we use the Fourier transformed variables \( \tilde{q}_k \) and \( \tilde{p}_k \) defined by

\[ q_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \tilde{q}_k e^{2\pi ink/N}, \quad p_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \tilde{p}_k e^{2\pi ink/N} \]

Show that these Fourier transformed variables are not Hermitian, but rather that \( \tilde{q}_{N-k}^\dagger = \tilde{q}_k \) and \( \tilde{p}_{N-k}^\dagger = \tilde{p}_k \).

(b) Write the Hamiltonian in terms of these Fourier transformed variables and show that it describes a set of \( N \) decoupled Harmonic oscillators, with frequencies you should compute. The excitations of these harmonic oscillators describe travelling waves on the crystal, and are known as phonons.

(c) Describe the Hilbert space of this theory in terms of the raising and lowering operators of these \( N \) SHOs. What is the spectrum of the Hamiltonian? This is a very simple (non-relativistic) one dimensional quantum field theory in the large \( N \) limit. Under what circumstances would it be appropriate to approximate the displacements \( q_n(t) \) by a continuous field \( \phi(x,t) \); for what observables would this be a good approximation?

3. Lagrangian Formulation of Electromagnetism

Electric and magnetic fields can be written as derivatives of two potentials, the scalar potential \( \Phi \) and the vector-potential \( \vec{A} \). It is convenient to replace the magnetic field with the antisymmetric 2-tensor \( F_{ij} \) defined as

\[ F_{ij} \equiv \partial_i A_j - \partial_j A_i \]

so that

\[ F_{ij} = \epsilon_{ijk} B_k \quad \text{and} \quad B_i = \frac{\epsilon_{ijk}}{2} F_{jk} \]

where Roman indices \( i,j,k \) run over 1,2,3 and we use Einstein summation notation. Here \( \epsilon_{ijk} \) is the totally antisymmetric symbol (Levi-Civita tensor) defined to be 1 when \( (ijk) \) is an even permutation of (123), \(-1\) when \( (ijk) \) is an odd permutation of (123) and zero otherwise.

(a) Write the electric and magnetic fields \( E_i \) and \( F_{ij} \) in terms of \( \partial_t \), \( \partial_i \), \( A_i \), and \( \Phi \).
(b) Now define \( A^\mu = (\Phi, \vec{A}) \) (where \( \mu = 0, 1, 2, 3 \) and the notation means that \( A^0 = \Phi \) and \( A^{1,2,3} = (\vec{A})^{1,2,3} \)). and define \( \partial_\mu = (\partial_t, \partial_i) \). Introduce
\[
F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu,
\]
and show how \( F^{0i} \) is related to the electric field and how \( F^{ij} \) is related to the magnetic tensor.

(c) Now, introduce the current 4-vector \( j^\mu = (\rho, \vec{j}) \). Show that
\[
\partial_\mu F^{\nu\mu} = [\pm] j^\nu
\]
is equivalent to both Gauss’ law and Ampere’s law. Figure out which is the correct sign on the current; is \( \pm \) a + or a −?

(d) Now define \( \epsilon_{\mu\nu\alpha\beta} \) the 4D antisymmetric symbol which generalizes the 3-D Levi-Civita tensor: \( \epsilon_{\nu\mu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta} \) and similarly for any other permutation of the indices, and \( \epsilon_{0123} = -1 \). [The minus sign is so that \( \epsilon^{0123} = +1 \); the sign flips because an odd number of the \( g^{\mu\nu} \)'s you need to raise the indices are negative.] Show that
\[
\epsilon_{\mu\nu\alpha\beta} \partial^\nu F^{\alpha\beta} = 0
\]
is an identity (is true regardless of what values \( A^\mu \) take provided they are twice differentiable) and that this identity is equivalent both to Gauss’ Law for magnetism and to Faraday’s law.

(e) Show that the the equations of motion of electromagnetism can be derived from the Lagrangian
\[
\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu
\]
where \( J^\mu \) is the current 4-vector.

(f) Show that Maxwell’s equations imply that this current four vector is conserved \( \partial_\mu J^\mu = 0 \). This is the usual continuity equation for charge and current \((\dot{\rho} = \vec{\nabla} \cdot \vec{j})\). Note that the conserved charge \( \int d^3x J^0 \) associated with this current is just the total charge.

(g) Show that \( F_{\mu\nu} \) is unchanged if we shift \( A_\mu \to A_\mu + \partial_\mu \Lambda \), where \( \Lambda \) is an arbitrary function of \( x^\mu \). This is known as a gauge transformation. Show that the action
\[
S_{EM} = \int d^4x \mathcal{L}_{EM}
\]
is invariant as well (up to a total derivative) only if the current is conserved.

4. Units

As in class, Roman indices refer to the three space indices 1,2,3 corresponding to \( x, y, z \) and Greek indices are 4-vector indices. It is essential to distinguish between upper and lower indices, since they are raised and lowered using the metric. For example, \( A_0 = A^0 \) but \( A_1 = -A^1 \).
In this class we will use units $\hbar = 1$ and $c = 1$. This means that all of the basic units are essentially the same

\[ length \propto time \propto (energy)^{-1} \propto (mass)^{-1} \]

We will usually use electron-Volts or centimeters. Compute the values of your 5 favourite (dimensionful) constants in units of cm and eV.