1. Complex Scalar Field

In this problem we will consider a complex scalar field $\phi$, with action

$$S_\phi = \int d^4x \left( \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right)$$

Unlike the real scalar in class, $\phi$ now assigns a complex (rather than real) number to each point in space-time. This is a long problem where you will really get to understand the quantization of free scalar fields in a very important case.

(a) Show that if we separate $\phi$ into its real and imaginary parts, $\phi = A + iB$, this action is completely equivalent to a pair of real free scalar fields with equal mass. Use this to argue that $\phi$ and its complex conjugate $\phi^*$ both solve the usual Klein-Gordon equation.

(b) It is convenient to think of $\phi$ and $\phi^*$ as independent fields, rather than the real and imaginary parts $A$ and $B$. Show that the Klein-Gordon equation following from variations of the action with respect to $\phi$ and $\phi^*$ are just the Klein-Gordon equations you wrote down above.

(c) Find the conjugate momenta, $\pi$ and $\pi^*$, to $\phi$ and $\phi^*$. Write down the canonical commutation relations.

(d) Show that the Hamiltonian is

$$H = \int d^3x \left( \pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \right)$$

(e) To quantize the theory, expand the field $\phi$ as

$$\phi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left( a_p e^{i\mathbf{p} \cdot \mathbf{x}} + b_p^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}} \right)$$

Note that, unlike the case in class, we are not assuming that $b_p^\dagger$ is the Hermitian conjugate of $a_p$. This is because $\phi$ is a complex, rather than real, field. Take the hermitian conjugate of this expression to write down the expansion of $\phi^*$ in terms of $a_p, a_p^\dagger, b_p, b_p^\dagger$. Assume that $a_p, a_p^\dagger, b_p, b_p^\dagger$ obey the usual SHO canonical commutation relations

$$[a_p, a_p^\dagger] = (2\pi)^2 \delta^{(3)}(\mathbf{p} - \mathbf{p}'), \quad [b_p, b_p^\dagger] = (2\pi)^2 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$$
Find the expansions for $\pi$ and $\pi^*$ in terms of the operators $a_p, a_p^\dagger, b_p, b_p^\dagger$ so that (given the SHO commutation relations above) the $\pi, \pi^*, \phi, \phi^*$ satisfy the canonical commutation relations of part (c). If you get confused, you can always use the real field $A$ and $B$ defined above.

(f) Compute the Hamiltonian in terms of the $a_p, a_p^\dagger, b_p, b_p^\dagger$. Show that the Hilbert space is exactly that of an infinite number of SHOs. In particular, conclude that you now have two different types of particles, one created by the $a_p^\dagger$ raising operator and the other created by the $b_p^\dagger$ operator.

2. Charged Particles

We will now continue to develop this theory to show that it is a model of charged particles. When we couple it to electromagnetism this will give scalar QED, which is a good approximation to the theory of electrons and photons if we neglect the fact that electrons have spin (and with some modifications it is a good model of superconductivity).

(a) Consider the complex scalar of the last problem. Show that the phase rotation $\phi \rightarrow e^{i\theta}\phi$ is a symmetry of the action. Compute the Noether current $J^\mu$ associated with this symmetry in terms of the fields $\phi$ and $\phi^*$. Show that the charge

$$Q = \frac{i}{2} \int d^3x \left( \phi^* \pi^* - \phi \pi \right)$$

is a constant in time.

(b) Rewrite this conserved charge $Q$ in terms of the raising and lowering operators $a_p, a_p^\dagger, b_p, b_p^\dagger$. What are the charges $Q$ of the following states $a_p^\dagger |0\rangle, a_p^\dagger a_p^\dagger |0\rangle, b_p^\dagger |0\rangle$? Compute the commutators of $a_p, a_p^\dagger, b_p, b_p^\dagger$ with $Q$. Argue that the particles created by $a_p^\dagger$ have positive charge and those created by $b_p^\dagger$ have negative charge; they are the anti-particles of one another. In scalar QED, these two types of particles would be the electrons and the positrons.

3. Higgs Field

With the addition of a second charged scalar, the theory is now has even more symmetry. This is the theory of a free Higgs boson; once we couple it to the appropriate generalizations of the electro-magnetic field (the electroweak gauge boson) and a potential it will describe electro-weak symmetry breaking in the standard model.

(a) Consider now the case where you have two complex scalar fields $\phi^a$, labelled by $a = 1, 2$, of the same mass.\(^1\) For each complex scalar there is a conserved charge $Q_a$, as above. Using the equations of motion, show that the following three charges are also conserved:

$$Q^i = \frac{i}{2} \int d^3x \left( \phi^*_a (\sigma^i)^a_b \pi^b - \pi^a (\sigma^i)^a_b \phi^*_b \right)$$

\(^1\)We will freely raise and lower the $a$ index, so that $\phi_a = \phi^a$, etc.
where \( i = 1, 2, 3 \) and \( \sigma^i \) is the usual Pauli matrix. Conclude that there are a total of four conserved charges, \( Q_1 + Q_2 \) as well as the \( Q^i \) described above.

(b) This theory now has four different types of particles, created by the \( a_{\alpha P}^\dagger, b_{\beta P}^\dagger \) where \( a = 1, 2 \). Write the three charges \( Q^i \) above in terms of these raising and lowering operators. Show that they obey the commutation relations of angular momentum (the \( SU(2) \) algebra):

\[
[Q^i, Q^j] = i\epsilon^{ijk} Q^k
\]

(c) The conserved quantities can also be understood as Noether charges of a symmetry. To see this, show that the action is now invariant under

\[
\begin{pmatrix}
\phi^1 \\
\phi^2
\end{pmatrix}
\rightarrow
U
\begin{pmatrix}
\phi^1 \\
\phi^2
\end{pmatrix}
\]

where \( U \) is a unitary 2 \( \times \) 2 matrix (i.e. obeys \( U^\dagger = U^{-1} \)).

(d) Show that, for any choice of (small) real parameters \( \epsilon^i \), the matrix \( U = 1 + i\epsilon_i \sigma^i + \ldots \) is a unitary matrix to leading order in the parameters \( \epsilon^i \). Write the first order variation of the fields \( \phi_a \) under the infinitesimal transformation parameterized by \( \epsilon^i \), and use Noether’s theorem to compute the corresponding conserved charges. You should find that (perhaps up to an overall constant) these are just the \( Q^i \) described above.

4. Fun with Euler-Lagrange Equations

Find the generalization of the Euler-Lagrange equation for a Lagrangian that is a function of the scalar field \( \phi \) and its first and second derivatives: \( \mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi) \).