Physics 610, Problem Set 4

due: Friday, October 9 at 3:30pm

Please place your completed problem sets in the “Physics 610” box in the physics department mailroom (Rutherford 103b) before the due date. Please do not leave them in my mailbox. You are encouraged to discuss these problems with your colleagues, but you must write up your own solutions; the solutions you hand in should reflect your own work and understanding. Late problem sets will be penalized 10% per day late, unless an extension has been obtained from me or the TA before the due date.

1. Lorentz Transformations

In class we argued that a general Lorentz transformation can be written as

$$\Lambda = \exp \{ i\omega_{\mu\nu} J^{\mu\nu} \}$$

where $\omega_{\mu\nu}$ is an antisymmetric matrix whose entries are the 6 rotation and boost parameters, and $J^{\mu\nu}$ is the $4 \times 4$ matrix of differential operators

$$J^{\mu\nu} = i(x^{\mu} \partial^{\nu} - x^{\nu} \partial^{\mu})$$

(a) Consider the case where $\omega_{01}$ is the only non-zero $\omega_{\mu\nu}$ parameter. How does $\Lambda$ act on the coordinates $x^{\mu}$? Verify that this is the Lorentz matrix is the usual boost in the $x$ direction, and determine how the parameter $\omega_{01}$ is related to the velocity $v$ of the boost.

(b) Verify that these generators obey the algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho})$$

This problem is much easier if you don’t try to compute all 36 commutators explicitly!

(c) The full Poincare symmetries of flat space also include the translations, which are generated by

$$P^{\mu} = -i \partial^{\mu}$$

Compute the commutators of these translation generators with the Lorentz generators defined above. Explain why these commutators mean that $P^{\mu}$ “transforms as a 4-vector” under Lorentz transformations.

(d) Define $J^{i} = \frac{1}{2} \epsilon^{ijk} J^{jk}$, the generator of rotations about the $i$-axis, and $K^{i} = J^{0i}$, the generator of boosts in the $i$-direction.

Rewrite the Lorentz algebra in terms of the $J$’s and $K$’s. Note that the commutator of two boosts is a rotation!

(e) Define

$$N^{i} = \frac{1}{2} (J^{i} + iK^{i}), \quad N^{i\dagger} = \frac{1}{2} (J^{i} - iK^{i})$$

and rewrite the Lorentz algebra in terms of the $N$’s and the $N^{\dagger}$’s. Conclude that the Lorentz algebra is two copies of the rotation algebra.
2. Lorentz Invariant Measure

The energy $\omega = \sqrt{\vec{p}^2 + m^2}$ and momentum $\vec{p}$ of a particle can be combined into an energy-momentum 4-vector $p^\mu = (\omega, \vec{p})$ which transforms in the usual way under Lorentz transformations

$$p^\mu \rightarrow \Lambda^\mu_\nu p^\nu$$

We will now understand why the integration measure we used earlier in the quantization of the free scalar field is Lorentz invariant.

(a) Show that

$$\int_{-\infty}^{\infty} dp^0 \delta(p^2 - m^2) \theta(p^0) = \frac{1}{\omega}$$

where $\theta$ is the usual Heaviside step function

(b) Show that the integration measure $d^4p$ is Lorentz invariant.

(c) Show that the integration measure $\frac{d^3\vec{p}}{2\omega}$ is Lorentz invariant.

3. Fun with the Clifford Algebra

In class we introduced matrices $\gamma^\mu$ which obey the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^\mu\nu$$

and used them to construct spinor representations of the Lorentz group. In this exercise you will study various properties of these matrices.

(a) Show the generators $S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$ obey the algebra of Lorentz transformations.

(b) Compute the commutator of $S^{\mu\nu}$ with $\gamma^\mu$. Compare this with your answer above for the commutator of the $J^{\mu\nu}$ with $P^\mu$. Explain why this means that $\gamma^\mu$ “transforms as a vector” under Lorentz transformations, i.e. that we can treat the $\mu$ index as a 4-vector index when we do a Lorentz transformation. Argue, for example, that the quantity $\gamma^\mu p_\mu$ is a Lorentz scalar, where $p_\mu$ transforms as a vector under a Lorentz transformation.

(c) Check that the gamma matrices defined in class

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

obey the Clifford algebra.

(d) Consider now a spinor $\psi$, which is a four-component object (you can think of it as a column vector) that transforms under a Lorentz transformation as

$$\psi \rightarrow (1 + i\omega_{\mu\nu} S^{\mu\nu} + \ldots) \psi$$

Here we are using the gamma matrices defined in part c. We have only written the transformation to linear order in $\omega_{\mu\nu}$.

Define $\bar{\psi} = \psi^\dagger \gamma^0$. How does $\bar{\psi} \psi$ transform to linear order in $\omega_{\mu\nu}$? Conclude that $\bar{\psi} \psi$ is a scalar under Lorentz transformations.

How does $\bar{\psi} \gamma^\mu \psi$ transform to linear order in $\omega_{\mu\nu}$? Conclude that $\bar{\psi} \gamma^\mu \psi$ is a vector under Lorentz transformations.