Physics 610, Problem Set 5

due: Friday, October 16 at 3:30pm

Please place your completed problem sets in the “Physics 610” box in the physics department mailroom (Rutherford 103b) before the due date. Please do not leave them in my mailbox. You are encouraged to discuss these problems with your colleagues, but you must write up your own solutions; the solutions you hand in should reflect your own work and understanding. Late problem sets will be penalized 10% per day late, unless an extension has been obtained from me or the TA before the due date.

1. Fermionic harmonic oscillator

Free scalar field theories reduce to a collection of uncoupled harmonic oscillators. These describe theories of bosons, meaning that the wave function doesn’t change when you exchange two identical particles; you can see this explicitly by noting that the two particle state $a_{p_1}^{\dagger}a_{p_2}^{\dagger}|0\rangle$ doesn’t depend on the ordering of the two creation operators.

Theories of fermions are built out of a collection of uncoupled Fermionic harmonic oscillators. The Fermionic oscillator is actually simpler than the usual SHO, but you may not be familiar with it. This problem will help you understand its properties.

The Fermionic oscillator is the quantum system with an operator $b$ which, together with its Hermitian conjugate $b^{\dagger}$, satisfy anti-commutation relations

$$\{b, b\} = 0 = \{b^{\dagger}, b^{\dagger}\}, \quad \{b, b^{\dagger}\} = 1$$

The Hamiltonian is

$$H = \frac{\omega}{2}(b^{\dagger}b - bb^{\dagger})$$

(a) Show that there must be a state annihilated by $b$. (Hint: what is $b^{2}\rangle$?) Name this (properly normalized) state $|0\rangle$.

(b) Argue that $b^{\dagger}|0\rangle$ cannot vanish. Call this state $|1\rangle$. Show that it is properly normalized.

(c) Now the punchline: show that the only states which you can find by acting with $b$ or $b^{\dagger}$ on $|0\rangle$ operators are linear combinations of $|0\rangle$ and $|1\rangle$. (Hint: Show that there are only 4 linearly independent operators which can be constructed from $b$ and $b^{\dagger}$.) Conclude that your Hilbert space is the usual 2-state system.

(d) What is the energy of each state?

2. Spin and Statistics

One of the deepest principles of quantum field theory is that particles with integer spin (scalars, photons, gravitons, etc.) have bosonic statistics and particles with half-integer spin (electrons, quarks, etc.) have fermionic statistics. This is a general consequence of Lorentz
invariance in four dimensions (it is actually not true in lower dimensions). We will not try
to prove this statement in general, but in this problem you will demonstrate explicitly that
scalars must be quantized with bosonic statistics if you want to get a sensible theory.

(a) Consider a free complex scalar field $\phi$, with action

$$\mathcal{L}_\phi = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

Compute the energy momentum tensor $T_{\mu\nu}$ and write down an expression for the energy
$E = \int d^3x T_{00}$ in terms of the field and its derivatives. Show that for any classical field
configuration $\phi(x)$ this energy is positive definite.

(b) Let’s now quantize the theory by expanding the field in terms of raising and lowering
operators

$$\phi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left( a_p^\dagger e^{ipx} + b_p e^{-ipx} \right)$$

with a similar expansion for $\phi^*$. Here $a_p^\dagger$ is the operator that creates a particle and $b_p^\dagger$ is
the operator that creates an anti-particle. In an earlier problem you verified that these
particles have opposite charge. Assume that these operators obey the usual commutation
relations

$$[a_p, a_q^\dagger] = (2\pi)^3 \delta^{(3)}(p - q), \quad [b_p, b_q^\dagger] = (2\pi)^3 \delta^{(3)}(p - q)$$

with all other commutators vanishing. Write the energy $E$ above in terms of the cre-
ation/annihilation operators. You should find that

$$E = \int d^3p \omega_p \left( a_p^\dagger a_p + b_p^\dagger b_p \right) + E_0$$

where $E_0$ is the energy of the ground state, which is unobservable and can (as usual) be
set to be equal to zero by adding an overall constant to the action. Show that with this
choice of ground state energy, the energy $E$ is positive definite in the quantum theory.

(c) Now repeat the above problem, except assuming that the particles now have Fermionic
statistics. This means that they obey

$$\lbrace a_p, a_q^\dagger \rbrace = (2\pi)^3 \delta^{(3)}(p - q), \quad \lbrace b_p, b_q^\dagger \rbrace = (2\pi)^3 \delta^{(3)}(p - q)$$

with all other anti-commutators vanishing. Show that the energy is now

$$E = \int d^3p \omega_p \left( a_p^\dagger a_p - b_p^\dagger b_p \right) + E_0$$

where $E_0$ is again the energy of the ground state (which we normalize to zero). Explain
why this means that particles and anti-particles now have opposite energy – particles
have positive energy, but anti-particles have negative energy – and that the energy is
unbounded below. Argue that when you turn on interactions which allow particles to be
produced, the theory will be unstable.
This turns out to be a completely general result – whenever you try to quantize a theory of scalars using anti-commutation relations (or to quantize a theory of spinors using commutation relations) you will find that the energy is unbounded below. In fact, one can also compute the two point function and show that it is not Lorentz invariant.

3. Spinor Identities

The plane wave solutions to the Dirac equation take the form

\[ \psi = u_s(p)e^{-ipx} \quad \text{and} \quad \psi = v_s e^{ipx} \]

Here \( s = 1, 2 \) labels the two linearly independent solutions.

(a) Consider a general frame where \( p^\mu = (E, \vec{p}) \). Write out the Dirac equation in the Weyl basis. Show that the Dirac equation is solved by \(^1\)

\[ u_s = \begin{pmatrix} \sqrt{\sigma \cdot p} \zeta_s \\ \sqrt{\sigma \cdot \bar{p}} \zeta_s \end{pmatrix}, \quad v_s = \begin{pmatrix} \sqrt{\sigma \cdot p} \eta_s \\ -\sqrt{\sigma \cdot \bar{p}} \eta_s \end{pmatrix} \]

where \( \zeta_s \) and \( \eta_s \) are constant two-component spinors. This is easiest if you don’t work this out explicitly, but instead use the fact (which you should prove) that

\[(p \cdot \sigma)(p \cdot \bar{\sigma}) = m^2\]

(b) Choose now a basis such that \( \zeta_s^\dagger \zeta_s = \delta_{rs} = \eta_s^\dagger \eta_s \). Show that this implies that \( \bar{u}^r u^s = 2m \delta^{rs} = \bar{v}^r v^s \) and \( \bar{u}^r v^s = 0 \).

(c) Show that if we sum over polarization states of the spinor we have

\[ \sum_s u_s(p) \bar{u}_s(p) = \gamma^\mu p_\mu + m \quad \sum_s v_s(p) \bar{v}_s(p) = \gamma^\mu p_\mu - m \]

Note that in this formula we are not writing explicitly the two spinor indices. For example, the first equation means

\[ \sum_s u_{sa}(p) \bar{u}_{sb}(p) = \gamma^\mu_{ab} p_\mu + m \]

where \( a, b = 1, 2, 3, 4 \) are spinor indices. We used this formula when computing the spinor propagator.

(d) Show that

\[ u_s(p) \gamma^\mu u_{s'}(p) = 2p^\mu \delta_{ss'} \]

This is a simple version of the Gordon identity.

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\(^1\)The square root of a matrix is defined by going to a basis where the matrix is diagonal and taking the square root of the eigenvalues.