1. **The Mass term of Free QFT as an Interaction**

Consider a free, real scalar field with Lagrangian \( \frac{1}{2}(\partial \phi)^2 - \frac{1}{2}m^2\phi^2 \). Let \( D^{m^2}(x_1, x_2) \) denote the Feynman propagator of a scalar field with mass \( m^2 \).

(a) So far in class we have regarded the mass term as part of the kinetic term – so, for example, we compute Feynman diagrams in this theory using the free propagator \( D^{m^2} \). But we could also regard this theory as a theory of a free massless scalar with an interaction term \( m^2\phi^2 \). Write down the position space Feynman rules for this theory, i.e. the Feynman rules using the massless propagator \( D^0 \) and a new (funny sort of) interaction vertex corresponding to \( m^2\phi^2 \).

(b) Draw the Feynman diagrams that contribute to \( \langle \phi(x_1)\phi(x_2) \rangle \) to order \( m^6 \), and evaluate them explicitly.

(c) Now imagine doing this to all orders in \( m^2 \). Sum the series and show that \( \langle \phi(x_1)\phi(x_2) \rangle \) is precisely the massive propagator \( D^{m^2} \), as expected.

2. **Feynmann Diagrams for Simple Integrals**

In this problem we will consider a toy version of version of scalar \( \phi^4 \) theory, where instead of computing a full path integral we compute a single integral:

\[
Z(\lambda) = \int_{-\infty}^{\infty} d\phi \exp \left\{ -\frac{1}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \right\}.
\]

where \( \phi \) is just a real number, rather than a field. This is what the path integral for scalar \( \phi^4 \) theory would look like if there were only one point in space-time (and if we went to Euclidean signature to make the integral convergent). A normal integral over a finite number of variables is therefore sometimes called “0-dimensional quantum field theory.”

We will just think of \( Z \) as a function of the coupling \( \lambda \). We could, if we wanted to, also include a “source term” \( j\phi \) in the integrand, and consider \( Z \) as a function of both \( j \) and \( \lambda \). We will keep \( j = 0 \), and just use this simple example to understand the convergence properties of the Feynman diagram expansion.
(a) Compute

- \( Z(0) \)
- \( Z(0.01) \)
- \( Z(0.1) \)
- \( Z(0.4) \)
- \( Z(1) \)
- \( Z(5) \)

numerically using your favourite numerical computing program (such as Maple or Mathematica).

(b) Let us now develop the Feynman diagram expansion for \( Z(\lambda) \). Replace \( \exp(-\lambda \phi^4/24) \) with its series expansion in \( \lambda \) (or equivalently, in \( \phi \)). Find the complete series expansion in \( \lambda \) in closed form, that is, write

\[
Z(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n, \tag{1}
\]

and find an explicit expression for \( c_n \). (Do this by expanding the exponent, exchanging orders of summation and integration, and doing the integral for each term in the series.)

(c) Explain how each \( c_n \) could be computed by using (appropriately modified) Feynman rules for our simple theory. Describe the Feynman rules for this theory and draw the appropriate "Feynman Diagrams" for the first few terms \( c_n \). Check that this matches the answer you got in part (b) for the first few \( c_n \). Note that, because we have not included the \( j\phi \) source term, there are no diagrams with external legs; we are computing the analog of "vacuum" diagrams in QFT.

(d) Evaluate the order \( \lambda^0 \) and \( \lambda^1 \) terms in this series numerically for the values of \( \lambda \) given in part (a). For which values of \( \lambda \) does the order \( \lambda^1 \) term help improve the accuracy of our perturbative expansion for \( Z(\lambda) \)?

(e) What happens to the integral \( Z(\lambda) \) when \( \lambda \) is negative? Conclude from your answer that the radius of convergence (in \( \lambda \)) of the series you developed in part (b) must be zero.

(f) Using Stirling’s approximation, find the asymptotic form of the \( c_n \) for large \( n \).

Now consider the individual terms \( \lambda^n c_n \) in the series for \( Z(\lambda) \). Show that, for fixed \( \lambda \), these terms will decrease as function of \( n \) until they reach a minimum at some critical value of \( n \) (call it \( n_0(\lambda) \)), after which they start increasing until they diverge at \( n \to \infty \). Use your approximate form for \( c_n \) to estimate \( n_0(\lambda) \) for small \( \lambda \). This shows explicitly that the radius of convergence of this series is zero.

Now, show that the smallest term in the series is

\[
\lambda^{n_0} c_{n_0} \sim e^{f(\lambda)}
\]

where \( f(\lambda) \) is a function you should determine. You need only determine the leading behaviour of \( f(\lambda) \) at small \( \lambda \).
(g) Based on this, you might conclude that the series in part (b) is useless. But that is not
the case at all; it still contains lots of useful information about $Z(\lambda)$!

The expansion

$$e^{-x} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} x^m = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots$$

has the following property for $x > 0$: the partial sums

$$f_n(x) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} x^m$$

are alternately strict over-estimates and strict under-estimates of the actual function; that is, for $x > 0$, $f_0(x) = 1 > e^{-x}$, $f_1(x) = (1-x) < e^{-x}$, $f_2(x) = (1-x+x^2/2) > e^{-x}$, and so forth with the $<,>$ alternating.

Use this property to show that the partial sums found above (i.e. Eq. (1) with $n$ cut off at
$0, 1, 2, 3, \ldots$), are alternately over-estimates and under-estimates of $Z(\lambda)$. Therefore, the true answer $Z(\lambda)$ always lies between neighbouring terms in the series of partial sums.

Use this property to find a bound for $Z(\lambda)$ at $\lambda = 1$, by evaluating alternating terms
until they start to diverge. How tight is the bound? Repeat for $Z(0.4)$ and $Z(0.1)$. How
does this compare with your numerical answer for $Z$?

Argue that, as $\lambda$ gets smaller and smaller, one can use the Feynman diagram expansion
to place tighter and tighter bounds on $Z(\lambda)$. Estimate for how tight the bound will be
(i.e. how big the error terms will be) as a function of $\lambda$, when $\lambda$ is small. You may find
the function $f(\lambda)$ in the previous problem useful.

We conclude that, while the series does not converge, it gives us very good information
about the value of $Z(\lambda)$. A series with this property – zero radius of convergence but the
ability to give good information near the origin – is called an asymptotic series. Feynman
diagram expansions in QFT typically only give asymptotic series, rather than convergent series.