No problem set for next week...

We are interested in computing $\langle f | i \rangle$

$$\langle f | i \rangle = (2\pi)^n \delta^{(n)}(\sum K_{in} - \sum K_{out})$$

$i M$ matrix element is computed:

1. Draw external lines for each $K_{in} \& K_{out}$
2. Connect these lines in all possible, topology really distinct ways using the interaction vertices \( (\phi^2 \Rightarrow \chi) \) 
   internal $q$ lines
3. Draw arrows on each line: $*$ incoming particle $\Rightarrow$ incoming arrow
   $*$ outgoing $\Rightarrow$ outgoing arrow
   internal lines $\Rightarrow$ arbitrary
4. Label each line by a momentum, enforce momentum conservation at each vertex:
   \[
   \begin{align*}
   p &\rightarrow p - K \\
   p &\leftarrow p + K
   \end{align*}
   \]
5. $\frac{\text{internal}}{p} = \frac{i}{p^2 - m^2 + i\varepsilon}$
   $\Rightarrow$ $\frac{\text{external}}{2\pi i}$ = 1

6. For an L-long dia, we will have L undetermined momenta
   \[
   \Rightarrow \text{integrate: } \int d^4l/(2\pi)^4 \\
   l^2 = m^2
   \]
"virtual particles" are not "on shell"

7. Divide by a symmetry factor: \( \times \) of symmetries under interchanging of internal prop. 4 vertices.

8. No bubble diags (completely disconnected from ext. legs)

No tadpoles (diags which become disconnected if you cut one line)

These rules allow for both connected & disconnected diags:

\[
\begin{align*}
2 \rightarrow 4 \text{ scattering:} & \quad \begin{array}{c}
\begin{array}{c}
\text{connected} \quad \text{diagrams} \\
\text{connected} \quad \text{diagrams}
\end{array}
\end{array} \\
& \quad \begin{array}{c}
\begin{array}{c}
\text{connected} \quad \text{diagrams} \\
\text{connected} \quad \text{diagrams}
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\text{connected} \quad \text{diagrams} \\
\text{connected} \quad \text{diagrams}
\end{array}
\end{array}
\end{align*}
\]

\[\delta^4(k_1 - k_2 - k_3) \delta^4(k_4 - k_5 + k_6)\]

\[\delta^4(k_1 + k_4 - k_2 - k_3 + k_5 - k_6)\]

\[\langle f | i \rangle \text{ connected} \propto \delta^4(k_1 - k_2 - k_3) \delta^4(k_4 - k_5 + k_6)\]

\[\langle f | i \rangle \text{ disconnected} \propto \delta^4(k_1 + k_4 - k_2 - k_3 + k_5 - k_6)\]

\[\text{Connected} \leftrightarrow \text{Disconnected} \]

\[\text{"Axiom"} \]

\[\text{Cluster Decomposition Principle:} \]

\[\langle \phi(x_1) \cdots \phi(x_n) \phi(x_{n+1}) \cdots \phi(x_{n+m}) \rangle = \langle \phi(x_1) \cdots \phi(x_n) \rangle \langle \phi(x_{n+1}) \cdots \phi(x_{n+m}) \rangle\]

\[\text{Long distance}\]

This is true if the expectation values are computed using local QFT

\[\begin{array}{c}
\begin{array}{c}
\text{connected} \quad \text{diagrams} \\
\text{connected} \quad \text{diagrams}
\end{array}
\end{array} \]

\[s_1 \quad s_2\]
\[ i \mathcal{M} = -\frac{i g^2}{4 \pi^2} \left( \frac{1}{(k_1 + k_2)^2 - m^2} + \frac{1}{(k_1 - k_3)^2 - m^2} + \frac{1}{(k_1 - k_4)^2 - m^2} \right) \]

The \( k_i \) are on-shell \( k_i^2 = m^2 \)

The amplitude has a pole when the internal particle is on-shell, \( (k_1 + k_2)^2 = m^2 \)

How many independent degrees of freedom are there in 2 \( \to \) 2 scattering? (after mom + E conservation)

CM frame \( \vec{k}_1 + \vec{k}_2 = 0 \) \( E_1 = \sqrt{\vec{k}_1^2 + m^2} = E_2 \)
\( \vec{k}_3 + \vec{k}_4 = 0 \) \( E_3 = E_4 \)

Energy cons \( \Rightarrow E_1 = E_3 \)

2 DOF: \( E_1 \) or \( |\vec{k}_1| \) and \( \Theta \)

Let's use Lorentz invariant quantities (sultos) to parameterize these DOF
Mandelstam variables
\[ s = (k_1 + k_2)^2 = 2m^2 + 2k_1 \cdot k_2 \]
\[ t = (k_1 - k_2)^2 = 2m^2 - 2k_1 \cdot k_2 \]
\[ u = (k_1 - k_4)^2 = 2m^2 - 2k_1 \cdot k_4 \]
\[ s + t + u = 6m^2 + 2k_1 \cdot (k_2 - k_3 - k_4) = 4m^2 \]

\[ \{ \begin{align*}
\text{cross} & = s \text{-channel} \\
\text{open} & = t \text{-channel} \\
\text{open} & = u \text{-channel}
\end{align*} \]

These variables can be used for 4 particle or dijitter mass:
\[ s + t + u = \sum m_i^2 \]

For 4 identical particles \( s > 2m^2 \)

Our scattering amplitudes:
\[ g \left( \frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{u - m^2} \right) \]

If the intermediate particle has mass \( m^2 \Rightarrow \) pole at \( s = m^2 \) is unobservable

Imagine a theory with 2 particles \( \phi \) and \( X \):
\( M^2 < \frac{1}{2} M_{\phi X}^2 \)

\[ \delta \phi \phi \phi \phi + \ldots \Rightarrow -ig^2 \left( \frac{1}{s - m_{\phi X}^2} + \frac{1}{t - m_{\phi X}^2} + \frac{1}{u - m_{\phi X}^2} \right) \]

has a pole at \( s = m_{\phi X}^2 \) which is visible
So far we've computed amplitudes, not experimental observables...

Consider a beam of particles

\[ \Phi = \text{Flux} = \frac{\# \text{ particles}}{\text{unit vol}} \times \text{velocity of beam} = \frac{\# \text{ of particles passing through area/ unit time}}{\text{target}} \]

For a fixed, hard target, you would observe the cross sectional area of the target

\[ \rightarrow \quad 0 \quad \rightarrow \quad \text{shadow} \]

We define the "cross section" of a scattering amplitude

\[ \sigma = \frac{\text{total \# particles scattered}}{\Phi \times \text{time}} \times \text{Area} = \pi r^2 \text{ for scattering from a hard sphere} \]

Typically we fix the initial momenta and consider \# particles scattered as a function of the final momenta

"Differential Cross section" \[\frac{d\sigma}{dS_1 \cdots dS_m}\]

\[dS_i = \text{solid angle into which particle } i \text{ is scattered} \]
e.g. in COM frame of 2→2 scattering $\frac{d\sigma}{d\Theta}$ - relative angle of initial \textit{i}th particle scattering

How do we compute $d\sigma$ from $\langle f | i \rangle$?

\[
\frac{N_{\text{scattered}}}{N_{\text{total}}} = \frac{P}{\langle f | f \rangle \langle i | i \rangle} = \frac{\langle f | i \rangle^2}{\langle f | f \rangle \langle i | i \rangle}
\]

\[
d\sigma = \frac{\langle f | i \rangle^2}{\langle f | f \rangle \langle i | i \rangle} \, d\Omega = \frac{\pi}{V} \, d^3p_i \downarrow \int d\Omega = 1
\]

\[
\Phi = \left| \frac{\mathbf{v}}{V} \right| \text{ in COM frame}
\]

\[
\Phi = \left| \frac{\mathbf{v} - \mathbf{v}_i}{V} \right|
\]

\[
d\sigma = \frac{1}{T \cdot \Phi} \, d\sigma
\]