\[ \alpha \rightarrow \beta = i \frac{(p + m)}{p^2 - m^2} \]

\[ \gamma_{\mu \nu} = -\frac{i}{p^2 + i\epsilon} g_{\mu \nu} \quad \text{Feyn. gauge} \]

\[ \gamma_{\mu} \gamma^{\alpha} = -i e \gamma^{\mu} \]

Contact spinor indices along lines of \( \rightarrow \) starting with a \( \bar{\psi} \) or a \( \bar{\psi} \) coming from outgoing \( e^- \) or \( e^+ \) and ending with a \( \bar{\psi} \) or a \( \bar{\psi} \) "in" or "out".

**Compton Scattering:** \( \bar{\psi} e^{-} \rightarrow e^+ \bar{\psi} e^- \)

\[ iM_5 = \left( ie \right)^2 \, \bar{\psi}_4 \gamma_\nu \gamma_\mu \psi_3 \, \frac{i \left( p_1 + p_2 + m \right)}{\left( p_1 + p_2 \right)^2 - m^2} \, \gamma_\mu \psi_2 \]

\[ iM_4 = \left( ie \right)^2 \, \bar{\psi}_4 \gamma_\nu \gamma_\mu \psi_3 \, \frac{i \left( p_3 - p_4 + m \right)}{\left( p_3 - p_4 \right)^2 - m^2} \, \gamma_\mu \psi_2 \]

We don't care about polarizations, so we need to sum over outgoing polarizations/spins and average over incoming...
\[ \sum_{\text{spins}} |M_+|^2 = \frac{e^4}{-m^2} \sum (\epsilon_1 \epsilon_2 \overline{\nu}_3 \gamma_\mu (\not{p_2} - \not{p_4} + m) \gamma_\nu \nu_2) (\overline{\nu}_2 \gamma_\mu (\not{p_2} - \not{p_4} + m) \gamma_\nu \nu_3 \epsilon_{1}^\alpha \epsilon_{2}^\beta) \]

\[ \Rightarrow \sum_S \overline{u}_2 \overline{u}_2 = p + m \quad \sum_{\text{pol.}} \epsilon_1 \epsilon_2 = -g^{\alpha \beta} \]

\[ \sum |M_+|^2 = \frac{e^4}{(4 - m^2)} \text{Tr} \left( \frac{1}{(\not{p_3} + m)(\not{p_2} - \not{p_4} + m)} \right) \]

\[ = \frac{e^4}{(4 - m^2)} \text{Tr} \left( \frac{1}{-2 \not{p_2} + 4m} \frac{1}{(\not{p_2} - \not{p_4} + m)} \right) \]

\[ = \frac{e^4}{(4 - m^2)} \left( \frac{1}{\not{p_3} \cdot \not{p_2}} + \frac{1}{\not{p_3} \cdot \not{p_4}} + \cdots \right) \]

Case of a photon scattering off an $e^-$ or $p^+$ at rest...

\[ p_2 = (w, 0, 0, 0) \quad p_1 = (w, 0, 0, w) \]

\[ p_3 = p_1 + p_2 - p_4 \quad p_8 = (w', w' \sin \theta, 0, w' \cos \theta) \]

\[ p_3^2 = w^2 = p_1^2 + p_2^2 + p_4^2 + 2p_1 \cdot p_2 + 2p_1 \cdot p_4 - 2p_2 \cdot p_4 \]

\[ \Rightarrow 0 = p_1 \cdot p_4 + p_2 \cdot p_4 - p_1 \cdot p_2 = w w' (1 - \cos \theta) + m w' - m w \]

\[ \Rightarrow \frac{w'}{w} = \frac{1}{1 + \frac{w}{m} (1 - \cos \theta)} \]

\[ \Delta \lambda = \frac{1}{w} - \frac{1}{w'} = \frac{1}{m} (1 - \cos \theta) \quad \text{"Compton formula"} \]
How does the intensity depend on $\Theta$?

\[ \Rightarrow \frac{1}{4} \sum_{\text{pol's}} |M|^2 \quad \text{from $M_5 + M_4$ above} \]

\[ \Rightarrow \frac{d\sigma}{d\cos \Theta} = \frac{\pi x^2}{m^2} \left( \frac{w'}{w} \right)^2 \left( \frac{w'}{w} + \frac{w}{w'} - \sin^2 \Theta \right) \]

"Klein-Nishina formula"

as $m \to \infty$ $w' \to w \Rightarrow \frac{d\sigma}{d\cos \Theta} \propto \frac{\pi x^2}{m^2} \left( 2 - \sin^2 \Theta \right) \]

"Thompson Scattering"
Rutherford Scattering: $e^- p^+ \rightarrow e^- p^+$

\[
\frac{d\sigma}{d\cos\theta} = \frac{4\pi e^4 m_e^2}{64\pi^2 p^4 \sin^2\frac{\theta}{2}} \left( 1 - \frac{v^2 \sin^2 \theta}{2} + \ldots \right)
\]

1st relativistic correction.

Classical Scattering.

In the limit $m_p \gg m_e$ so the proton will remain at rest:

- $p_2 = p_4 = (m_p, 0, 0, 0)$
- $p_1 = (E, 0, 0, p)$
- $p_3 = (E, 0, \sin \theta \, p, \cos \theta \, p)$

$\frac{v}{E} \ll 1$

QED.
Renormalization

This class I have avoided loops. They typically diverge...

e.g. \( L = -\frac{1}{2} \phi \nabla^2 \phi \phi - \frac{\lambda}{4!} \phi^4 \) massless \( \phi^4 \)

2 \( \rightarrow \) 2 scattering

\[ i \mathcal{M} = X + \begin{array}{c} p_1 \k_1 \k_2 \k_3 \k_4 \end{array} \]

\[ i \mathcal{M}_1 = -i \lambda \]

\[ i \mathcal{M}_2 = (-i \lambda)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2} \frac{i}{(p-k)^2} \]

\[ \int \frac{d^4 k}{k^4} \phi \approx \log \Lambda \] at large \( k \) (high energy or small distances)

\[ i \mathcal{M}_2 \approx -\log \frac{s}{\Lambda^2} \]

Computation: \( M_2(s) = -\frac{\lambda^2}{2} \log \frac{s}{\Lambda^2} \approx \text{cut} + \text{finite} \)

\( M_2(s) \) appears to be UV divergent, when written in terms of the coupling \( \lambda \) appearing in the action.

\( \lambda \) is supposed to measure the strength of interaction
How would we measure $\lambda$? You could define $\lambda$ in terms of $2\to2$ scattering at some fiducial energy $s_0$:

$$\lambda_{\text{R}} \equiv -M(s_0) = \lambda + \frac{\lambda^2}{32\pi^2} \log \left( \frac{s_0}{\Lambda^2} \right) + \ldots$$

$\lambda$ in $L$ is a "bare" coupling. It is unobservable. Only $\lambda_{\text{R}}$ is observable.

$M(s)$ in terms of $\lambda_{\text{R}}$:

$$M(s) = -\lambda - \frac{\lambda^2}{32\pi^2} \log \frac{s}{\Lambda^2}$$

$$M(s) = -\lambda_{\text{R}} - \frac{\lambda^2_{\text{R}}}{32\pi^2} \log \frac{s}{s_0} + \ldots$$

When we write $M(s)$ in terms of physical data rather than unobservable parameters, no divergences...

...general phenomenon.