Infrared Divergences

Many QFT scattering amplitudes have IR divergences when there are massless particles...

These arise b/c in theories w/massless particle there is no distinction between

\[ 1 \text{ e}^- \approx 1 \text{ e}^+ + \text{"soft" photon} \approx 1 \text{ e}^+ \text{ cloud of "soft" photon} \]

The S-matrix defined in terms of n-particle states is not well defined or observable.

Claim: Physical Observables, such as cross-sections, is finite.

Consider $\phi^3$ theory. When $m^2 \to 0$,

\[ T \to \frac{T_{\text{tree}}}{T_{\text{loop}}} \text{ IR divergent} \]

\[ T = \frac{\text{tree}}{\text{loop}} \]

\[ O(g^0) \quad O(g^2) \]

One of the external states with momentum $k$ could split into a 2-particle state

\[ T_{\text{split}} = \frac{\text{tree}}{\text{loop}} = \frac{i g}{k^2 + m^2} - i g \]

As $m \to 0$ this $T_{\text{split}}$ can diverge at $k^2 \to 0$.

The IR divergence in $T$ at 1-loop cancel the tree level contribution to $T_{\text{split}}$. 
Consider a detector. It can distinguish between a particle with momentum $K \neq \mathbf{0}$ and a pair $(K_1, K_2)$ s.t. $K_1 + K_2 = K$ and either $K_1$ or $K_2$ are very small.

A physical cross section must include a sum over all indistinguishable states.

$$|T_{\text{obs}}|^2 \frac{d^3 K}{E_0} = |T|^2 \frac{d^3 K}{E_i} + |T_{\text{split}}|^2 \frac{d^3 K_1 d^3 K_2}{E_1 E_2} + \ldots$$

$$|T_{\text{obs}}|^2 = |T|^2 \left( 1 + \frac{g^2}{(k^2 + m^2)^2} E \delta^3 (K - K_1 + K_2) \frac{d^3 K_1}{E_1} \frac{d^3 K_2}{E_2} + \ldots \right)$$

$$= |T_{\text{tree}}|^2 + g^2 (T_{\text{tree}} T_{\text{1-loop}} + c.c.) + g^2 E \delta^3 (K - K_1 - K_2) \frac{d^3 K_1}{E_1} \frac{d^3 K_2}{E_2}$$

since $T = T_{\text{tree}} + g^2 T_{\text{1-loop}} + \ldots$

$$|T_{\text{1-loop}}|$$ is IR divergent

the correction term from $T_{\text{split}}$ is IR divergent as well:

As $m^2 \to 0$, $K^2 = (k_1 + k_2)^2 = E_1 E_2 \sin^2 \theta_2 \propto E_1 E_2$ at fixed $\Theta$

$$\frac{d^3 K_1}{E_1} \sim E_1 dE_1$$

since $|K_1| = E_1$,

$$\frac{d^3 K_2}{E_2} \sim E_2 dE_2$$

$$\Rightarrow g^2 \frac{1}{k^4} E \frac{d^3 K_1}{E_1} \frac{d^3 K_2}{E_2} \propto \frac{dE_1 dE_2}{E_1 E_2}$$ is IR divergent.
Diagrammatically, we can see why these cancel...

\[ \text{Tree } T_{\text{loop}}^2 \quad \text{vs.} \quad |T_{\text{split}}|^2 \]

\[ \begin{array}{c}
\text{"cut graph notation"}
\end{array} \]

\[ \begin{array}{c}
\text{"cut"}
\end{array} \]

\[ |T_{\text{split}}|^2 \approx \begin{array}{c}
\text{these two contributions are represented by the same cut graph}
\end{array} \quad \Rightarrow \quad \text{they cancel.} \]

One can prove that IR divergences cancel at all orders...

At the end of the day, we will continue to consider S-matrix elements, introducing an \( \Lambda \) regulator if necessary.
Remarkability of QED

Summary: So for all UV divergences in QED are resolved if we write things in terms of observable quantities.

\[ L = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \bar{\psi} \left( i \gamma^\mu - e A^\mu - m \right) \psi \]

in terms of bare fields & couplings

\[ \Rightarrow \quad -\frac{1}{4} Z_3 F_{\mu \nu}^2 + i Z_2 \bar{\psi} \gamma^\mu \gamma^\nu \psi + Z_2 Z m \bar{\psi} \psi - e r \bar{\psi} A^\mu \psi \]

\[ \Rightarrow \quad \psi \quad \text{We can expand} \quad Z_i = 1 + \delta_i \quad \text{counterterms.} \]

\[ \Rightarrow \quad \psi \quad -\frac{1}{4} F_{\mu \nu}^2 + i \bar{\psi} \gamma^\mu \gamma^\nu \psi + Z m \bar{\psi} \psi - e r \bar{\psi} A^\mu \psi \]

\[ \Rightarrow \quad \psi \quad \text{We can compute in pert. theory using} \quad Z m \text{ &} \quad e r \text{ as our coupling constants in Feyn. Dings.} \]

\[ \Rightarrow \quad \psi \quad \text{Fix} \quad \delta_i \quad \text{by imposing physical conditions...} \]

\[ i G(p) = \quad \text{1PI contributions.} \]

\[ \Rightarrow \quad \Sigma(m_R) = 0 = \Sigma'(m_R) \quad \text{fixes} \quad \delta_2, \delta_3 \]

Similarly \[ i G_{\mu \nu} = m \delta_{\mu \nu} = -i \left( g_{\mu \nu} - \frac{\epsilon_{\mu \nu} \pi}{p^2} \right) \]

\[ \Pi(0) = 0 \quad \text{fixes} \quad \delta_3 \]

\[ p^2 + p^2 \pi(p^2) + \text{const} \]
\(-i e \Gamma^\mu(p) = \Gamma^\mu(p)\) since non-1PI can be interpreted as connection to \(\Pi\) or \(\Sigma\)

\[
\Gamma^\mu = F_\mu(p^2) \gamma^\mu + \frac{i \sigma^\mu \nu p_\nu}{2m} F_\nu(p^2)
\]

\[
\Gamma^\mu(\rho, 0) = \gamma^\mu, \text{ fixes } \delta,
\]

We have 4 physical conditions

\[
\begin{align*}
\Sigma(m_R) &= 0 \\
\Sigma'(m_R) &= 0 \\
\Gamma^\mu(0) &= \gamma^\mu \\
\Pi(0) &= 0
\end{align*}
\]

for 4 unknowns \(\delta_i\).

\[
\text{In Dim Reg, } \delta_2 = \frac{e^2}{8\pi^2} \left(-\frac{1}{2} \left(-\gamma + \log\frac{m^2}{4\pi}\right) + \frac{1}{2} \log m_R^2 - \frac{5}{2} - \log \frac{m_R^2}{m^2} + \cdots\right)
\]

\[
\delta_m = \frac{e^2}{8\pi^2} \left(-\frac{3}{2} \left(-\gamma + \log\frac{m^2}{4\pi}\right) + \frac{3}{2} \log m_R^2 - \frac{5}{2} + \cdots\right)
\]

the form of the \(\delta_i\) depend on Reg. Scheme.

The physical conditions \(\Sigma(m_R) = 0, \Sigma'(m_R) = 0\) are just a choice. \(\Rightarrow\) choice that \(m_R\), the parameter in \(\Sigma\), to be the physical mass.

\(\Rightarrow m_R = m_p\) is "on-shell scheme".

Other schemes can be used to make the \(\delta_i\) simpler...

Minimal Subtraction \((MS)\): \(\delta_i = \) divergent piece but no finite piece
\[ S_2 = -\frac{e^2}{8\pi^2} \frac{1}{\varepsilon} \quad S_{\mu} = -3 \frac{e^2}{8\pi^2} \frac{1}{\varepsilon} \]

In this scheme, the propagator has a pole at \( \not{\mu} = m_p \) where \( m_p \neq M_p \)

\[ G = \frac{1}{\not{\mu} - m_p + \Sigma} \]

\[ \Sigma(m_p) = M_p - m_p \]

\[ \Sigma'(m_p) = m_p \]

In MS we could compute the reln between \( m_p \) and \( m_p \)

\[ MS_s = S_2 + \text{divergent piece} + \gamma + \log 4\pi \]

with no other finite piece.

\( m_0, e_0, \ldots \)

\( \Delta m, e, \ldots \)

\( m_p, e_p \)

\( \Lambda = \text{UV cutoff scale} \)

\( \mu = \text{"subtraction" scale where we impose physical cond.} \)

Only \( m_p, e_p \) are physically observable.

Q: Are these 4 content enough? Can all UV divergences be cancelled just by adjusting these 4 counterterms?

A: Yes.

A theory such that all UV divergence can be cancelled by a finite set of counterterms is renormalizable.