A beautiful (but incorrect) alternative to the theory of weak interactions was proposed by Georgi and Glashow. In this theory, the $SU(2) \times U(1)$ electroweak gauge sector is replaced by a single $SU(2)$ gauge group, with action

$$L = -\frac{1}{2} D^\mu \phi D_\mu \phi - V(\phi) - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

Here $a = 1, 2, 3$ is an $SU(2)$ index and the field $\phi$ is in the adjoint representation, so that

$$D^\mu \phi = \partial_\mu \phi + e \epsilon^{abc} A^b_\mu \phi^c$$

The potential

$$V(\phi) = \frac{1}{8} \lambda (\phi^a \phi^a - v^2)^2$$

is such that the vacuum state breaks $SU(2)$ symmetry.

1. Show that a vacuum solution with $\phi^a \phi^a = v^2$ is a minimum energy solution to the equation of motion, and breaks $SU(2)$ gauge symmetry down to $U(1)$. For a given vacuum solution $\phi^a$, which of the $A^a_\mu$ is massless and which are massive? The massless $U(1)$ describes standard electromagnetism. Compute the mass $m_W$ of the massive $W$-bosons in the theory. What are the charges of the $W$-bosons?

2. We are interested in finding a soliton solution of the theory. A soliton is a static (i.e. time-independent) solution to the equations of motion which describes a localized lump of energy. Basically, solitons look like new types of particles which are built out of excitations of $\phi$. We will start in this problem by looking for solutions to the equations of motion with only the scalar field turned on, so that $A^a_\mu = 0$.

(a) Show that a static solution $\phi^a(x)$ to the equations of motion will have energy $E = T + U$, where $T = \int d^3 x |\nabla \phi^a|^2$ and $U = \int d^3 x V(\phi^a)$ are the contributions to $E$ from gradient and potential energy, respectively.
(b) Assume that we have a soliton solution $\phi^a(x)$ with gradient energy $T$ and potential energy $U$. Show that the total energy $E(\alpha)$ of the solution $\phi^a(x/\alpha)$ is $\alpha T + \alpha^3 U$.

(c) Argue that $\frac{dE}{d\alpha} = 0$ at $\alpha = 1$.

(d) Use this to prove that no soliton solution can exist. This is a version of Derrick’s Theorem, which states that any theory with only scalar fields has no soliton solutions in space-time dimension greater than 2. In fact, in last semester’s final you found a soliton solution (called a “kink”) in a two dimensional scalar theory.

3. This means that any soliton solution must have the gauge field turned on as well. So let us now look for a static, finite energy solution $A_\mu^a(x)$, $\phi^a(x)$.

(a) Show that the energy of such a solution (which we will call $M$, since it’s the mass of the soliton) is

$$M = \int d^3x \left( \frac{1}{2} E_i^a E_i^a + \frac{1}{2} B_i^a B_i^a + D_i \phi^a D_i \phi^a + V(\phi) \right)$$

where

$$E_i^a = F_{0i}^a, \quad B_i^a = \frac{1}{2} \epsilon_{ijk} F_{jk}^a$$

are non-abelian generalizations of electric and magnetic field. Here $i = 1, 2, 3$ is a spatial index, and repeated indices are summed over as usual.

(b) Consider a solution of the form

$$\phi^a(r) = f(r)x^a/r, \quad A_i^a = a(r)\epsilon^{a_1a_2a_3}x_j^{a_1}x_j^{a_2}/r^2$$

where $a = 1, 2, 3$. Assume that $f(r) \to f(\infty)$ and $a(r) \to a(\infty)$ approach constants at $r \to \infty$. Determine $f(\infty)$ and $a(\infty)$ by demanding that the soliton has finite mass.

Show that as $r \to \infty$ the scalar field must approach a minimum of the potential $V(\phi)$, i.e. $\phi^a \phi^a = v^2$, but that for different values of $x^a/r$ (i.e. at different locations on the sphere at spatial infinity) we are at different locations on the minimum energy surface.

(c) Explain why we must have $f(0) = 0$. This means that in the core of the soliton – near $r = 0$ – the gauge symmetry is unbroken and all of the gauge bosons are massless.

(d) By investigating the form of $A_i^a$ at large $r$, argue that the solution has magnetic charge. (By magnetic charge, we mean that the solution carries magnetic charge under the unbroken $U(1)$ gauge field of electromagnetism). So this soliton is a magnetic monopole, called the ’T hooft-Polyakov monopole.

(e) Compute the magnetic charge $g$ of the soliton and show that it satisfies the Dirac Quantization Condition

$$4\pi eg = n,$$

for some integer $n$ which states that magnetic charge times electric charge must be quantized.
The existence of a magnetic monopole would provide a lovely explanation for the observed quantization of electric charge. Unfortunately, the standard model – unlike the Georgi-Glashow model – does not have magnetic monopole solutions, because the $U(1)$ of electromagnetism is not contained in an $SU(2)$. In grand unified theories, however, $U(1)_{EM}$ lives inside a larger $SU(2)$ gauge group which is broken at the GUT scale. So GUT theories will contain magnetic monopoles and naturally explain the quantization of charge.

4. We have not yet proven that these monopole solutions actually exist, by solving for $a(r)$ and $f(r)$.

(a) Write out explicitly the equations of motion for $\phi^a$ and $A_i^a$, and use this to write a set of differential equations for $a(r)$ and $f(r)$.

(b) Show that in the limit $\lambda \to 0$ (so that we can neglect the potential $V$) these equations are solved by

$$a(r) = a(\infty) - \frac{b}{\sinh evr} r, \quad f(r) = f(\infty) \coth evr - \frac{c}{evr}$$

where $b$ and $c$ are numbers you should compute. You have found the magnetic monopole, known as the Bogomolny-Prasad-Sommerfeld (BPS) solution!

(c) Compute the mass of the monopole and show that

$$M = \kappa \frac{m_W}{e^2}$$

where $\kappa$ is a constant which you should compute. Compute the magnetic monopole mass in a grand unified theory, when $m_W$ is taken to be the GUT scale, about $10^{15}$ GeV.

Note that when the field theory is weakly coupled ($\epsilon$ is small) the monopole is very heavy. This fits our natural view that the magnetic monopole is a composite particle built out of excitations of the perturbative field $\phi$. However, when $\epsilon$ becomes large (so that the theory is strongly coupled) the monopole becomes light. One can imagine that when $\epsilon$ is very large in fact the monopole should be regarded as the perturbative object and the field $\phi$ should be a composite field built out of a monopoles. In fact, in many situations this is precisely correct, and leads to a phenomenon known as duality: our quantum field theory in fact has two completely equivalent descriptions. In one, the electric coupling $\epsilon$ is small and the field $\phi$ is viewed as the fundamental object out of which a monopole can be built, whereas in the other the magnetic coupling $g$ is small and the monopole field is viewed as the fundamental object out of which electrically charges objects can be built.