The atomic force microscope (AFM) is an instrument that measures the spatial variation of local fields by sensing the deflection of a micrometer-sized cantilever. The fields are electromagnetic in origin and interact with the atoms on the surface of another object, referred to as the "sample." The deflection of the cantilever necessitates the field doing work on the cantilever. In this paper we discuss the minimum mass (typically \(10^{-11}\) kg) that responds to the coupling of gravitational radiation from a distant astrophysical source such as a pulsar or supernova. Oscillations in the Riemann tensor field exert a force on the antenna. The resonant mechanical excitation of the resonant bar is detected by a very low-noise displacement sensor, typically a capacitively or inductively coupled superconducting quantum interference device. The magnitude of the excitation is expected to less than \(10^{-19}\) m. The AFM cantilever is a very small mass (typically \(10^{-11}\) kg) which is made to couple with electromagnetic fields. The resulting excitation is most often detected by an optical sensor. Since the displacements typically sensed by the AFM are relatively large, on the order of \(10^{-9}\) m, the forces can often be sensed nonresonantly. Alternatively, the coupling field between the tip and sample can be made time varying at the resonant frequency of the lever. In this case an oscillating force excites the resonant mode of the cantilever. A variation on this detection scheme involves sensing field gradients by monitoring the shift of the lever resonant frequency. In resonant operation the amplitude of the fundamental mode of the AFM cantilever is usually measured by a phase-sensitive detector, as common in GWA detection. A GWA is a resonant force sensor of very large dimension, typically \(10^{14}\) times heavier than the micromechanical AFM lever. This means that a GWA operating at the same resonant frequency as an AFM needs a detector with \(10^7\) times greater displacement sensitivity in order to measure the same amount of energy (since \(\langle (\Delta x)^2 \rangle = 2AEo/mo^2\)). This in turn places stringent requirements on the electromagnetic transducer, a problem not usually encountered with the AFM. The fundamental sensitivity of both the AFM and the GWA depends on the same parameters since both are damped harmonic oscillators in contact with a heat bath. There are three fundamental noise sources common to both techniques: intrinsic noise of the detector, the backaction forces of the detector on the oscillator, and the Brownian motion of the oscillator. The minimum detectable energy, the result of the uncertainty principle, is

\[
\Delta E_{\text{QM}} = 2hf_0,
\]

where \(h\) is Planck's constant and \(f_0\) is the resonant frequency of the oscillator. This corresponds to the energy required to create two phonons in the oscillator. For a cantilever with \(f_0 = 10\) kHz, \(\Delta E_{\text{QM}} = 1.3 \times 10^{-29}\) J.

Although a variety of detectors have been used, the AFM typically uses an optical system to detect the deflection of the cantilever, either an interferometer or, more commonly, an optical lever. Tunnel transducers, although they have extremely high sensitivity, suffer from a variety of practical problems related to the small working distance and contamination of the tunnel gap. Capacitive transducers not only have the problem of lack of sensitivity due to the small size of the oscillator but also are limited by the reciprocal back-action of the electronic amplifier. It has been shown that the ideally focused and illuminated optical lever has the same signal-to-noise ratio as the Michelson interferometer, i.e., it depends on the wavelength, the power, and the Poisson distribution of the light source. The light source can have fluctuations in intensity, phase (in the case of the interferometer), or beam direction (for the optical lever), but these effects can be minimized by proper experimental design. The total displacement noise can therefore be expressed as

\[
\langle (\Delta x)^2 \rangle = \langle (\Delta x)^2 \rangle_{\text{shot}} + \langle (\Delta x)^2 \rangle_{\text{back}},
\]

where the first term is the result of photon shot noise at the photodiode and the second term, the backaction of the light on the cantilever, is the result of fluctuations in the radiation pressure. The displacement noise for an ideal (lossless) Michelson interferometer has been calculated for an interferometric GW detector. The configuration of an interferometer-based AFM is somewhat different (see e.g., Ref. 9), but the physical principles are the same. The radiation pressure fluctuations cause a fluctuating force

\[
\langle (\Delta P)^2 \rangle_{\text{back}} = 8Phf/c\lambda,
\]
FIG. 1. A gravitational wave antenna absorbs energy from an astrophysical event via coupling with the Riemann curvature tensor. An atomic force microscope lever absorbs energy from an atomic process via coupling with the Maxwell stress tensor. Both antenna and lever are harmonic oscillators whose motion is monitored by a low-noise displacement sensor.

where \( \lambda \) is the wavelength, \( P \) is the optical power of the light at the lever,\(^{10} \) and \( \Delta f \) is the measurement bandwidth. In the nonresonant case, much below \( f_0 \),

\[
\langle (\Delta x)^2 \rangle_{\text{back}} = (8P\Delta f/c\lambda)(1/k)^2
\]

and at the resonance

\[
\langle (\Delta x)^2 \rangle_{\text{back}} = (8P\Delta f/c\lambda)(Q/k)^2,
\]

where \( k \) is the cantilever spring constant and \( Q \) is its mechanical quality defined by the decay time \( \tau_0 = Q/\pi f_0 \). The shot noise contribution is\(^{6,10} \)

\[
\langle (\Delta x)^2 \rangle_{\text{shot}} = h c \lambda \Delta f / 16 \pi^2 P \cos^2(\varphi/2),
\]

where \( \varphi \) is the phase difference between the two optical paths in the interferometer. Since the AFM interferometer is normally operated in quadrature (i.e., \( \varphi = \pi/2 \)) for maximum displacement sensitivity,

\[
\langle (\Delta x)^2 \rangle_{\text{shot}} = h c \lambda \Delta f / 8 \pi^2 P
\]

and

\[
\langle (\Delta x)^2 \rangle_{\text{shot}} = [c\lambda/8 \pi^2 P + (8P/c\lambda)(Q/k)^2]h \Delta f.
\]

The expression for \( \langle (\Delta x)^2 \rangle \) can be minimized to find the optimum optical power at resonance (for off-resonance, set \( Q = 1 \)):

\[
P_0 = k c \lambda / 8 \pi Q.
\]

This can be used to find the minimum displacement noise

\[
\langle (\Delta x)^2 \rangle_{\text{min}} = 2hQ\Delta f / \pi k.
\]

The minimum detectable energy is therefore

\[
\Delta E_{\text{min}} = (k/2)(\langle (\Delta x)^2 \rangle_{\text{min}}) = hQ\Delta f / \pi.
\]

If we assume the smallest bandwidth is given by \( \Delta f = 2/\tau_0 = 2\pi f_0/Q \), then we find that

\[
\Delta E_{\text{min}} = 2h f_0,
\]

which is the quantum limit. The geometry of the optical lever is more complicated to analyze in detail, but it is presumed that these results apply to it with only slight modification.

For the relatively low powers typically used with the AFM (1–100 \( \mu \)W), the optical deflection sensors are usually shot-noise limited, not backaction limited, with displacement sensitivities between \( 10^{-12} \) and \( 10^{-15} \) m/Hz. However, using soft levers with high \( Q \), the backaction forces can become dominant. In vacuum, silicon cantilevers have been reported with \( Q \) as high as \( 6.2 \times 10^4 \) at room temperature.\(^{3} \)

Using the parameters \( \lambda = 670 \) nm, \( k = 0.2 \) N/m, and \( Q = 5 \times 10^4 \), we find that \( P_0 = 32 \) \( \mu \)W and \( \langle (\Delta x)^2 \rangle_{\text{min}} = 0.9 \times 10^{-14} \) m/Hz. Therefore, if one wants to use a softer cantilever to achieve greater force sensitivity, it is in principle necessary to lower the optical power. Otherwise, radiation pressure fluctuations will dominate the sensitivity (assuming for the moment that there are no other noise sources). The laser power can be reduced until Johnson noise in the photodetector becomes significant. The existence of an optimal optical power also means that for a given power there is an optimal mass of the cantilever

\[
m_0 = (2Q/\pi f_0^2)(P/c\lambda).
\]

For \( P = 10 \) \( \mu \)W, \( \lambda = 670 \) nm, \( Q = 5 \times 10^4 \), and \( f_0 = 10 \) kHz, we find that \( m_0 = 1.6 \times 10^{-11} \) kg, which is in the range of microfabricated cantilevers.\(^{\text{11}} \)

The effect of including losses in the optical detector is to increase the backaction force relative to the shot noise. This occurs because light is lost before it arrives at the photodetector, thereby increasing the shot noise, but after it has exerted a force on the cantilever. In addition, light absorbed by the cantilever may exert additional thermal forces. The number of photons absorbed by the cantilever raises its temperature and may cause a constant deflection due to thermal expansion effects, analogous to the constant deflection due to radiation pressure. But, as with radiation pressure, it is only a fluctuation in the number of photons absorbed which causes a fluctuation in temperature and thereby a fluctuation in displacement. For homogeneous silicon nitride cantilevers, radiation pressure fluctuations dominate radiation-induced temperature fluctuations in an optical AFM detector.\(^{\text{11}} \) In contrast, for gold-coated silicon nitride cantilevers (which show a bimetallic effect) the radiation-induced temperature fluctuations cause a deflection approximately 90 times greater than the radiation pressure fluctuations.\(^{\text{11}} \) This suggests that inhomogeneous levers, such as those used for the magnetic force microscope\(^{\text{13}} \) or for the micromechanical chemical sensor,\(^{\text{12}} \) may need to be operated at significantly lower powers in order to achieve optimum detector sensitivity.

The thermal noise force due to the lever being in contact with a heat bath at temperature \( T \) is\(^{\text{13}} \)

\[
m_0 = (2Q/\pi f_0^2)(P/c\lambda).
\]
\[
\langle (\Delta F)^2 \rangle_{\text{therm}} = 2k_B T \Delta f / \pi f_0 Q.
\]

For AFM experiments performed at the resonance frequency, the thermal noise of the cantilever is greater than noise originating from the detector or from backaction forces. The thermal displacement noise at the cantilever resonance is

\[
\langle (\Delta x)^2 \rangle_{\text{th}} = 2k_B T Q \Delta f / k \pi f_0
\]

and the total displacement noise is

\[
\langle (\Delta x)^2 \rangle = \left[2k_B T Q / k \pi f_0 + (8P h / c \lambda)(Q / k)^2 + h c / 8 \pi^2 P \right] \Delta f.
\]

For \(k=0.2 \text{ N/m}, f_0=10 \text{ kHz} \) and \(Q=5 \times 10^4\), the thermal noise amplitude at room temperature is \(2.6 \times 10^{-10} \text{ m/Hz}^1\), many orders of magnitude above the noise levels for the other two contributions. Even at atmospheric pressure where \(Q\) is typically around 50, the thermal noise at resonance is on the order of \(10^{-11} \text{ m/Hz}\). If we assume that the AFM is operated at its optimum optical power as defined above, then

\[
\langle (\Delta x)^2 \rangle = \langle (\Delta x)^2 \rangle_{\text{th}}^0 + \langle (\Delta x)^2 \rangle_{\text{min}}^0 = \left[2k_B T Q / k \pi f_0 + 2h Q / \pi k\right] \Delta f.
\]

Thus a 10 kHz lever operating with optimum illumination is detector limited, and thereby also quantum limited, only at a temperature below 0.5 \(\mu\text{K}\).

Resonant AFM operation means that a continuous sinusoidal signal, such as a time-varying electric or magnetic field, is tuned precisely to the lever resonant frequency. For weak signals a long integration time is therefore used to extract the coherent signal from the thermal noise. The GWA analogy is the detection of continuous gravitational radiation from pulsars and binary stars. As mentioned above, resonant-mode AFM measurements will typically be dominated by thermal fluctuations. Since the signal force is amplified at resonance by a factor of \(Q\), the smallest detectable force is

\[
\Delta F_{\text{th}} = k \langle (\Delta x)^2 \rangle_{\text{th}}^{1/2} = (4k_b T / Q \pi f_0 \tau_s)^{1/2},
\]

where \(\tau_s = 2/\Delta f\) is the integration time used in the experiment. The minimum detectable energy is

\[
\Delta E_{\text{th}} = (\Delta F_{\text{th}})^2 / 2k - 2k_b T / Q \pi f_0 \tau_s.
\]

Assuming \(\tau_s = \tau_0 = Q / \pi f_0\) then

\[
\Delta F_{\text{th}} = (4k_b T / Q)^{1/2},
\]

and

\[
\Delta E_{\text{th}} = 2k_b T / Q^2.
\]

The minimum detectable power is

\[
\Delta P_{\text{th}} = \Delta E_{\text{th}} / \tau_s = 2k_b T / Q^2.
\]

For the parameters above where \(\tau_s = \tau_0 = 1.6 \text{ s},\)

\[
\Delta F_{\text{th}} = 1.2 \times 10^{-15} \text{ N}, \quad \Delta E_{\text{th}} = 3.3 \times 10^{-28} \text{ J}, \quad \text{and} \quad \Delta P_{\text{th}} = 2.1 \times 10^{-30} \text{ W}.
\]

When the AFM is used to sense force gradients, one monitors the resonant frequency \(\omega_0 = k_{\text{eff}} / m\) where \(k_{\text{eff}} = k + \partial F / \partial z\). To detect a gradient, the position of the lever must be modulated. For a small amplitude of lever oscillation \(\Delta z\), the measured force modulation is \(\Delta F_z = (\partial F / \partial z) \Delta z\), where it is assumed that \(\partial F / \partial z\) does not vary appreciably over the distance \(\Delta z\). Thus, the minimum detectable force gradient, limited by thermal noise, is

\[
\Delta F_{\text{th}} = \Delta F_{\text{th}} / \Delta z_{\text{osc}} = (4k_b T / Q \pi f_0 \tau_s)^{1/2} / \Delta z_{\text{osc}},
\]

where \(\Delta z_{\text{osc}}\) is the rms amplitude of the forced cantilever oscillation.

The zero temperature limit for the force and energy resolution is

\[
\Delta F_0 = (4k_b h / \pi f_0)^{1/2} / Q
\]

and

\[
\Delta E_0 = 2h \pi f_0 \tau_s,
\]

which for the condition \(\tau_s = \tau_0\) gives

\[
\Delta F_0 = (4k_b h f_0)^{1/2} / Q
\]

and

\[
\Delta E_0 = 2h f_0 / Q^2.
\]

The fact that signals with an average energy of less than one phonon can be detected is the result of the integration procedure. This means only that the integrated value of very low power signals can be compared to the thermal energy \(2k_b T\) or the quantum energy \(2h f_0\).

As shown above, when the AFM is thermally dominated the minimum detectable force \(\Delta F_{\text{th}}\) and energy \(\Delta E_{\text{th}}\) depend on \(T/Q^2\). The value of \(T/Q^2\) can be considerably improved to yield much higher sensitivities. In vacuum at a temperature of 4.2 K, the \(Q\) of a silicon micromachined cantilever has been measured to be \(7.5 \times 10^5\). For \(f_0=10 \text{ kHz}\) and \(k=0.2 \text{ N/m}\) we find \(\Delta F_{\text{th}}=9.1 \times 10^{-21} \text{ N}, \Delta E_{\text{th}}=1.2 \times 10^{-34} \text{ J}, \text{ and} \Delta P_{\text{th}}=5.0 \times 10^{-36} \text{ W}.

The factor \(T/Q^2\) emphasizes that doubling \(Q\) is equivalent to reducing \(T\) by a factor of 4, although \(Q\) normally increases as \(T\) decreases. Also, because \((\Delta F_{\text{th}})^2 / k T / Q^2\) the tradeoff between \(k\) and \(Q\) should be carefully considered since thinner, and therefore softer, levers usually have lower \(Q\) because of the increased surface to volume ratio.

It should be pointed out that it is possible to decrease \(\tau_0\) (i.e., to increase the bandwidth) while keeping \(T/Q\) constant by the process of "cold damping." This has the advantage of allowing shorter measurement times as well as making it easier to tune the oscillator to the external coherent signal. This technique has been used for many years with continuous wave GWA and has also recently been applied to the AFM.

There are a few force microscopy experiments that attempt to reach the ultimate resolution limits discussed above, in particular the experiments that measure magnetic resonance using a force microscope. The present force sensitivity in these room temperature experiments is \(5 \times 10^{-16} \text{ N}^2\). Since the energy of a magnetic moment in a magnetic field is \(E=-\mu_m B\), the force exerted is \(F_z=-\partial E / \partial z = \mu_m \partial B / \partial z\). Let us assume that it is possible to make a cantilever with \(k=2 \times 10^{-3} \text{ N/m}\) while keeping \(f_0=10 \text{ kHz}\) and \(Q=7.5 \times 10^5\). At a temperature of 4.2 K,
$\Delta F_{th}=9.1\times 10^{-19}$ N. Since the magnetic moment of an electron is $\mu_B=\frac{e\hbar}{2mc}=9.27\times 10^{-24}$ J/T, a field gradient of $\partial B/\partial z=10^5$ T/m is required to detect a single electron spin. For a proton the constraints on $\Delta F_{th}$, $\partial B/\partial z$ or the number of spins are increased by the factor $g_\mu/m_\mu/g_\mu m_\mu = 658$.

For frequencies well below $f_0$, the thermal displacement noise amplitude is

$$\langle (\Delta x)^2 \rangle_{th}=2k_B T f/k\pi f_0 Q.$$  

For nonresonant operation, the backaction forces are negligible for the optical powers used in the AFM. The total displacement noise is then

$$\langle (\Delta x)^2 \rangle_{th}+\langle (\Delta x)^2 \rangle_{shot}$$

$$=[2k_B T/k\pi f_0 Q + h c/8\pi^2 P] \Delta f.$$  

For soft levers ($k=0.2$ N/m, $f_0 = 10$ kHz) at atmospheric pressure ($Q=50$) and room temperature, the thermal noise amplitude is $1.6\times 10^{-13}$ m/Hz while the shot noise amplitude of a 10 µW, 670 nm optical detector is $1.3\times 10^{-14}$ m/Hz. When thermal noise is dominant,

$$\Delta F_{th}=k\langle (\Delta x)^2 \rangle_{th}^{1/2}-(2k_B T \Delta f/Q \pi f_0)^{1/2}$$

and

$$\Delta E_{th}=1/2(\Delta F_{th})^2/k=k_B T \Delta f/Q \pi f_0,$$

which is the same result as for the resonant case. If we again assume that the smallest reasonable bandwidth is $\Delta f=2/\tau_0=2\pi f_0/\Omega$, then we again find that

$$\Delta F_{th}=(4k_B T)^{1/2}/Q$$

and

$$\Delta E_{th}=2k_B T/\Omega^2.$$  

For the parameters above, $\Delta F_{th}=1.2\times 10^{-12}$ N and $\Delta E_{th}=3.3\times 10^{-24}$ J with $\Delta f=1.3$ kHz. The measurable power is $\Delta P_{th}=2.1\times 10^{-21}$ W.

A silicon lever in vacuum with $Q=5\times 10^4$ has a thermal noise amplitude of $5.1\times 10^{-12}$ m/Hz, while a silicon nitride lever, with $Q=2\times 10^5$, has a thermal noise level of $1.6\times 10^{-14}$ m/Hz. Since most AFM detectors have shot-noise limited sensitivities between $10^{-13}$ and $10^{-14}$ m/Hz, nonresonant-mode AFM in vacuum can usually be considered detector limited. Assuming $\langle (\Delta x)^2 \rangle_{th}^{1/2}=10^{-14}$ m/Hz, $k=0.2$ N/m, and $\Delta f=1$ kHz, the detector-limited force resolution is $\Delta F_{det}=6.3\times 10^{-16}$ N and the corresponding energy resolution is $\Delta E_{det}=10^{-26}$ J.

In most AFM experiments performed in air the tip is actually in contact with the sample so that the end of the cantilever is not free to oscillate under the influence of thermal forces. The lowest vibrational mode when the tip is in contact has a resonance frequency $4.8$ times that of the free cantilever and a spring constant 4 times greater. Since $\langle (\Delta x)^2 \rangle_{th}$ depends on $(k f_0)^{-1}$, the thermal oscillation amplitude will be reduced by about a factor of 20 when the tip touches the sample. An interferometer-based AFM, which senses motion near the end of the cantilever, will find the thermal noise to be negligible. The angular displacement, to which the optical lever is sensitive, will be reduced by about a factor of 10. Thus, a lever with $k=0.2$ N/m, $f_0=10$ kHz, and $Q=50$ will show a thermal noise amplitude of approximately $10^{-14}$ m/Hz in the contact mode. Contact measurements can therefore typically be considered detector limited.

The weakest chemical bonds, van der Waals interactions, have energies of $1$ kJ/mol = $1.66\times 10^{-21}$ J/molecule, while the strongest ionic bonds have energies about 500 times larger. As shown above, $\Delta E_{th}$ in an ambient AFM experiment is on the order of $10^{-24}$ J. That this sensitivity is possible makes sense since in order for the AFM to measure the atomic-scale topography of surfaces nondestructively, which it apparently can do,2 it must be able to measure forces between atoms and molecules without breaking any chemical bonds.

With the AFM the experimenter has in principle complete freedom over the time scale of the experiment. For example, in resonant-mode operation it is possible to adjust the frequency of the drive signal to exactly that of the cantilever resonance and to average for as long as all parameters can be kept constant. In nonresonant operation it is similarly possible to scan over the surface at a frequency as slow as desired and to make an appropriate filter. Normally the measurement will be limited only by ambient vibrations and 1/f noise. But it is also possible with the AFM to detect single burst events which are nonrepeating and where a long averaging time would lower the signal to noise ratio. Such a signal might, for example, be the result of individual molecular events. A short sampling time (large bandwidth) is required to extract the signal from the noise since the amount of energy in the burst is to be compared with the thermal fluctuations in the same time interval. The situation is analogous to the detection of a burst of gravitational radiation with a resonant bar antenna. Under these circumstances the thermal fluctuations in the cantilever give a sensitivity limit

$$\Delta E_{th}=2k_B T (\tau_0/\tau_s) = (2k_B T/\Omega) (\pi f_0 \tau_s).$$

The ratio $(\tau_0/\tau_s)$ shows that the thermal fluctuations of the oscillator within the sampling time $\tau_s$ are reduced as its relaxation time $\tau_0=Q/\pi f_0$ increases. At room temperature with $f_0=10$ kHz, $Q=5\times 10^4$ and $\tau_0=10$ ms, $\Delta E_{th}=5.2\times 10^{-23}$ J, again much smaller than the weakest chemical bonds.

The limit on the sampling time is the detector noise. The AFM will be maximally sensitive at $f_0$ with a usable bandwidth determined by the detector noise. If we assume that the detector is operated at its optimum optical power, an optimal bandwidth can be found by minimizing

$$\Delta E_{tot}=\Delta E_{th}+\Delta E_{min}=2k_B T f_0 \tau_s/Q + 2h \Omega/\pi \tau_s$$

with respect to $\tau_s$. This makes the thermal noise and detector noise have equal value. The result is an optimal sampling time

$$\tau_{os}=\left(Q/\pi\right)\left(h f_0 k_B T\right)^{1/2}$$

and

$$\Delta E_{burst}=4(h f_0 k_B T)^{1/2}.$$  

For $T=4.2$ K, $f_0=10$ kHz and $Q=3.5\times 10^5$, $\tau_{os}=8.1$ ms, and $\Delta E_{burst}=7.8\times 10^{-20}$ J. The bandwidth is $\Delta f=2/\tau_s=250$ Hz and the effective noise temperature is $5.6$ mK. We can define
\[ \langle n \rangle = \Delta E_0/hf_0 = 4(k_B T/hf_0)^{1/2}, \]
which describes the number of phonons in the cantilever. For the low temperature parameters the number of detectable phonons is \( \langle n \rangle = 1.2 \times 10^4 \). For a spring constant of 0.2 N/m it should be possible to measure individual burst forces at the level of \( 1.8 \times 10^{-13} \) N.

In summary, it has been demonstrated that the AFM is a device capable of quantum-limited energy detection. For nonresonant operation the AFM is often limited by detector noise, while for resonant operation thermal noise is dominant. As is the case of GWAs, achieving lower temperatures and fabricating cantilevers with higher \( Q \)'s will increase the sensitivity of AFM. The advances in experimental GWA have been impressive over the last 20 years, and it is reasonable to expect that AFM development will follow a similar course. AFM has the additional advantage that it is much easier to cool to ultralow temperatures the small mass of the AFM lever than the large mass of the GWA. In addition, ultralow-loss materials such as silicon and sapphire are much easier to obtain in smaller sizes (although very small masses generally have lower \( Q \) due to the increased surface to volume ratio).

I wish to acknowledge discussion of this topic with Mark Bocko and Dan Rugar, and I thank Gerd Binnig for preparing the drawing in Fig. 1.

References:
10. In Ref. 8 the optical power considered is the input power to the interferometer, which is twice the power reflecting from a single mirror and twice the power arriving at the photodiode. In the AFM we consider the power reflected at the lever, which is usually a rather small percentage of the input power.