UV Divergences in Maximal and Half-Maximal Supergravities

K.S. Stelle

Adventures in Superspace Workshop
in Honor of Marc Grisaru

McGill University
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G. Bossard, P.S. Howe & K.S.S. 0901.4661, 0908.3883, 1009.0743
G. Bossard, P.S. Howe, K.S.S. & P. Vanhove 1105.6087
G. Bossard, P.S. Howe & K.S.S., 1212.0841
Marc and the origins of Superspace nonrenormalization theorems

- Marc was a pioneer in the application of S-matrix amplitude methods to the analysis of supergravity infinities.
  
  *Grisaru, Pendleton & Van Nieuwenhuizen 1977*

- He also pioneered the application of the background field method in the analysis of infinities.
  
  *Abbott, Grisaru & Schaefer 1983
  Grisaru, van Nieuwenhuizen & Wu 1975*

- These techniques were put to use in a dramatic way in the calculation of 4-loop beta functions for supersymmetric sigma models.
  
  *Grisaru, van de Ven & Zanon 1986*

- He also introduced the key nonrenormalization theorem for extended supersymmetry.
  
  *Grisaru & Siegel 1982*
- Key tools in proving non-renormalization theorems are superspace formulations and the background field.

- For example, the Wess-Zumino model in N=1, D=4 supersymmetry is formulated in terms of a chiral superfield $\phi(x, \theta, \bar{\theta})$:
  $$\bar{D}\phi = 0 \quad ; \quad \bar{D}_\alpha = -\frac{\partial}{\partial \bar{\theta}^\alpha} - i\theta^\alpha \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}. $$

- In the background field method, one splits the superfield into “background” and “quantum” parts,
  $$\phi = \phi + Q$$
  background    quantum

- The chiral constraint on $Q(x, \theta, \bar{\theta})$ can be solved by introducing a “prepotential”:
  $$Q = \bar{D}^2 X \quad (\bar{D}^3 \equiv 0)$$
• Although the Wess-Zumino action requires chiral superspace integrals 
  \[ I = \int d^4x d^4\theta \bar{\phi} \phi + \text{Re} \int d^4x d^2\theta \phi^3 \]
  when they are written in terms of the total field \( \phi \), the parts involving the quantum field \( Q \) appearing inside loop diagrams can be re-written as 
  \[ \int d^4x d^4\theta = \int d^4x d^2\theta d^2\bar{\theta} \]
  full-superspace integrals using the “integration = differentiation” property of Berezin integrals.

• Upon expanding into background and quantum parts, one finds, \textit{e.g.}, that the chiral interaction terms can be re-written as full superspace integrals:
  \[ \int d^4x d^2\theta Q^2 \phi = \int d^4x d^4\theta X \bar{D}^2X \phi \]

• Thus all counterterms written using the background field \( \Phi \) must be writable as full-superspace integrals.

\textit{Grisaru, Siegel & Rocek 1979}
• The degree of “off-shell” supersymmetry is the maximal supersymmetry for which the algebra can close without use of the equations of motion.

• Knowing the extent of this off-shell supersymmetry is tricky, and may involve formulations (e.g. harmonic superspace) with infinite numbers of auxiliary fields.

• For maximal N=4 Super Yang-Mills and maximal N=8 supergravity, the fraction of off-shell realizable supersymmetry is known to be at least half the full supersymmetry of the theory, but the maximum realizable fraction in harmonic superspace is not currently known. Assuming that the maximal fraction is 1/2 lead originally to the expectation that the first allowable counterterms would have “1/2 BPS” structure.
- The 3-loop $R^4$ maximal supergravity candidate counterterm has a structure under linearized supersymmetry that is very similar to that of an $F^4$ N=4 super Yang-Mills invariant. Both of these are 1/2 BPS invariants, involving integration over just half the corresponding full superspaces:

$$\Delta I_{SYM} = \int (d^4 \theta d^4 \bar{\theta})_{105 \, \text{tr}(\phi^4)_{105}}$$

$$\Delta I_{SG} = \int (d^8 \theta d^8 \bar{\theta})_{232848 (W^4)_{232848}}$$

- Versions of these supergravity and SYM counterterms indeed do occur at one loop in D=8. This implies that, at least in that spacetime dimension, the full nonlinear structure of such counterterms exists and is consistent with all symmetries.

<table>
<thead>
<tr>
<th>Loop order</th>
<th>Dimension</th>
<th>Gen. form</th>
<th>BPS degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>105</td>
<td>$\phi_{ij}$</td>
<td>6 of $SU(4)$</td>
</tr>
<tr>
<td>2</td>
<td>232848</td>
<td>$W_{ijkl}$</td>
<td>70 of $SU(8)$</td>
</tr>
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</table>

Deser, Kay & K.S.S. 1977 (N=1 case)

Howe, K.S.S. & Townsend 1981
Kallosh 1981
Unitarity-based calculations

- The calculational front has now made substantial progress since the late 1990s.

- This has led to unanticipated and surprising cancellations at the 3- and 4-loop orders, yielding new lowest possible orders for the super Yang-Mills and supergravity divergence onset.

plus 46 more topologies

Max. SYM first divergences, current lowest possible orders (for integral spacetime dimensions).

Max. supergravity first divergences, current lowest possible orders (for integral spacetime dimensions).

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<td>3</td>
<td>6?</td>
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Blue: known divergences
The construction of supersymmetric invariants is isomorphic to the construction of cohomologically nontrivial closed forms in superspace: \( I = \int_{M_0} \sigma^* L_D \) is invariant (where \( \sigma^* \) is a pull-back to a section of the projection map down to the purely bosonic “body” subspace \( M_0 \)) if \( L_D \) is a closed form in superspace, and it is nonvanishing only if \( L_D \) is nontrivial.

Using the BRST formalism, one can handle all gauge symmetries including space-time diffeomorphisms by the nilpotent BRST operator \( s \). The invariance condition for \( L_D \) is

\[
s L_D + d_0 L_{D-1} = 0 ,
\]

where \( d_0 \) is the usual bosonic exterior derivative. Since \( s^2 = 0 \) and \( s \) anticommutes with \( d_0 \), one obtains using Poincaré’s lemma

\[
s L_{D-1} + d_0 L_{D-2} = 0 ,
\]

etc.
Cohomological non-renormalization theorem

- Counterterm cohomology then allows one to derive non-renormalization theorems: the cocycle structure of a candidate counterterm and its associated operators must match that of the classical action.

  - In maximal SYM, this leads to a non-renormalization theorem ruling out the $F^4$ counterterm that was otherwise expected at $L=4$ in $D=5$.

  - Similar non-renormalization theorems exist in supergravity, but their study is complicated by local supersymmetry and the density character of counterterm integrands.
Duality invariance constraints

- Maximal supergravity has a series of duality symmetries which extend the automatic $\text{GL}(11-D)$ symmetry obtained upon dimensional reduction down from $D=11$. The classic example is $E_7$ in the $N=8$, $D=4$ theory, with the $70=133-63$ scalars taking their values in an $E_7/\text{SU}(8)$ coset target space.

- The $N=8$, $D=4$ theory can be formulated in a manifestly $E_7$ covariant (but non-manifestly Lorentz covariant) formalism. Anomalies for $\text{SU}(8)$, and hence $E_7$, cancel.

- Combining the requirement of continuous duality invariance with the superspace cohomology requirements gives further powerful restrictions on counterterms.

Other approach to duality analysis from string amplitudes:
- Broedel & Dixon 2010
- Elvang & Kiermeier 2010;
- Beisert, Elvang, Freedman, Kiermaier, Morales & Stieberger 2010

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Supergravity Densities

- In a curved superspace, an invariant is constructed from the top (pure “body”) component in a coordinate basis:

\[ I = \frac{1}{D!} \int d^D x \, \varepsilon^{m_D \ldots m_1} \, E_{m_D} A_D \ldots E_{m_1} A_1 \, L_{A_1 \ldots A_D}(x, \theta = 0) \]

- Referring this to a preferred “flat” basis and identifying \( E_M^A \) components with vielbeins and gravitinos, one has, e.g. in D=4

\[
I = \frac{1}{24} \int \left( e^a e^b e^c e^d \, L_{abcd} + 4 e^a e^b e^c \psi^\alpha L_{abc\alpha} + 6 e^a e^b \psi^\alpha \psi^\beta \, L_{ab\alpha \beta} \right.
\]

\[
+ 4 e^a \psi^\alpha \psi^\beta \psi^\gamma L_{a\alpha \beta \gamma} + \psi^\alpha \psi^\beta \psi^\gamma \psi^\delta L_{\alpha \beta \gamma \delta} \bigg) \]

Thus the “soul” components of the cocycle also contribute to the local supersymmetric covariantization.

- Since the gravitinos do not transform under the D=4 E\(_7\) duality, the \( L_{ABCD} \) form components have to be separately duality invariant.
At leading order, the E\(_7\)/SU(8) coset generators of E\(_7\) simply produce constant shifts in the 70 scalar fields. This leads to a much easier check of invariance than analyzing the full superspace cohomology problem.

Although the pure-body (4,0) component \(L_{abcd}\) of the \(R^4\) counterterm has long been known to be shift-invariant at lowest order (since all 70 scalar fields are covered by derivatives), it is harder for the fermionic “soul” components to be so, since they are of lower dimension.

Thus, one finds that the maxi-soul (0,4) \(L_{\alpha\beta\gamma\delta}\) component is not invariant under constant shifts of the 70 scalars. Hence the D=4, N=8, 3-loop \(R^4\) 1/2 BPS counterterm is not E\(_7\) duality invariant, so it is ruled out as an allowed counterterm.
L=7 and Vanishing Volume

- The above type of analysis knocks out all the candidates in D=4, N=8 supergravity through L=6 loops. This leaves 7 loops (Δ=16) as the first order where a fully acceptable candidate might occur, with the volume of superspace as a prime candidate: \( \int d^4 x d^{32} \theta E(x, \theta) \).

- Explicitly integrating out the volume into component fields using the superspace constraints implying the classical field equations would be an ugly task.

  However, using an on-shell implementation of harmonic superspace together with a superspace implementation of the normal-coordinate expansion, one can in fact evaluate it, but one then finds that the volume vanishes:

  \[ \int d^4 x d^{32} \theta E(x, \theta) = 0 \text{ on-shell} \]
1/8 BPS $E_7$ invariant candidate notwithstanding

- Despite the vanishing of the full $N=8$ superspace volume, one can nonetheless use an on-shell harmonic superspace formalism to construct a different manifestly $E_7$-invariant but 1/8 BPS candidate:

$$I^8 := \int d\mu_{(8,1,1)} \ B_{\alpha\dot{\beta}} \ B^{\alpha\dot{\beta}}$$

$$B_{\alpha\dot{\beta}} = \bar{\chi}_{\dot{\beta}}^{1ij} \ \chi^\alpha 8ij$$

- At the leading 4-point level, this invariant of generic $\partial^8 R^4$ structure can be written as a full superspace integral with respect to the linearized $N=8$ supersymmetry. It cannot, however, be rewritten as a non-BPS full-superspace integral with a duality-invariant integrand at the nonlinear level.

- Non-BPS full-superspace and manifestly $E_7$-invariant candidates do exist in any case from 8 loops onwards.
The N=4 Supergravity L=3 surprise

- Not everything is perfect in the understanding of supergravity divergences, however. A surprize has occurred in an unexpected sector: D=4, N=4 supergravity at L=3. The expected 3-loop $R^4$ divergence ($\Delta=8$) does not occur in that theory. \textit{Bern, Davies, Dennen & Huang 2012}

Yet, the L=7 candidate counterterm of N=8 supergravity has a natural analogue here as a 1/4 BPS $(4,1,1)$ G-analytic invariant: $I^4 = \int d\mu_{(4,1,1)} B_{\alpha\dot{\beta}} B^{\alpha\dot{\beta}} B_{\alpha\dot{\beta}} = \chi_{1}^\alpha \bar{\chi}_{\dot{4}}$ \textit{Bossard, Howe, K.S.S. & Vanhove 2011}

Expanding the content of this N=4 invariant at linearized level, one finds a leading $R^4$ structure undressed by the $SL(2, \mathbb{R})/U(1)$ complex scalar field: it is perfectly duality invariant, just like the 1/8 BPS candidate 7-loop N=8 counterterm. \textit{Bossard, Howe, K.S.S. & Vanhove 2011}
Vanishing volumes and their consequences

- Another aspect of this story needs to be clarified. The vanishing of a superspace volume can open the door to another representation of candidate counterterms.

- Consider the cases where superspace volumes vanish on-shell:

  ◆ The full superspace volumes of all D=4 pure supergravities vanish, for any extension N of supersymmetry.

  ◆ In D=5, the volume of maximal (32 supercharge) supergravity does *not* vanish, but the volume of half-maximal (16 supercharge, i.e. N=2, D=5) supergravity *does*.
Half-maximal D=5, L=2

- Unitarity-based calculations in D=5 half-maximal supergravity show cancellation of $R^4$ divergences at the 2-loop level similar to those found in half-maximal D=4, L=3.

- This cancellation is equally surprising as in the N=4, D=4 case, because there is an available 1/4 BPS D=5 (4,1) G-analytic $Sp(2)/(U(1) \times Sp(1))$ counterterm:

$$\int d\mu_{(4,1)} \Omega^{\alpha\beta} \Omega^{\gamma\delta} \left( x^1_x x^1_{\beta} x^1_\gamma x^1_\delta \right)$$

where $\Omega^{\alpha\beta}$ is the D=5 Lorentz $Sp(1,1)$ symplectic matrix.

- Moreover, in D=5 there are no complications from anomalies to the “duality” shift symmetry for the single scalar $\phi$ of half-maximal D=5 supergravity, unlike the D=4, N=4 case.
• The vanishing volume of half-maximal D=5 supergravity invites another way to write a candidate $\Delta=8$ counterterm in D=5. One can write simply

$$I^{4'} = \int d^{16} \theta E \Phi$$

where $\Phi$ is the D=5 field-strength superfield containing the scalar $\phi$ as its lowest component field.

• Also, this candidate is clearly invariant under the rather minimalistic D=5 duality symmetry $\Phi \rightarrow \Phi + \text{constant}$, since $\int d^{16} \theta E = 0$.

• Moreover, this candidate turns out to be just a rewriting of the above (4,1) G-analytic manifestly duality invariant 1/4 BPS candidate counterterm.

• In this sense, the D=5 $\Delta=8$ (4,1) $R^4$ counterterm is of marginal F/D type.
• The D=4 (4,1,1) G-analytic counterterm has the same marginal F/D character.

• The D=4, N=4 theory has as lowest-dimension physical component a complex scalar field \( \tau \) taking its values in the Kähler space \( SL(2,\mathbb{R})/U(1) \). In terms of \( \tau \), the Kähler potential is

\[
K[\tau] = -\ln(\text{Im}[\tau])
\]

and the N=4, \( \Delta=8 \) (4,1,1) counterterm can equally well be written

\[
\int d^{16}\theta E K[\tau]
\]

• As in the D=5 case, although this full-superspace integral is duality invariant, its integrand is not duality invariant. The integrand varies as follows:

\[
\delta (E \ln(\text{Im}[\tau])) = 2hE + fE(\tau + \bar{\tau})
\]
Superspace nonrenormalization theorems: refinement of the duality invariance requirement

- The marginal F/D structure of the $\Delta=8$ counterterm candidates in half-maximal $D=4$ and $D=5$ supergravities requires a more careful treatment of the Ward identities for duality.

- If one makes the assumption that there exist off-shell full 16-supercharge superfield formulations for the half-maximal theories, then one can derive a stronger requirement for duality invariance: not only must the integrated counterterm be invariant, but also the counter-Lagrangian superfield integrand must itself be duality invariant.

- Proof of this refined theorem requires introduction of the notion of a chain of superspace co-forms arising from the duality variation of the Lagrangian density. In order for the duality Ward identities to be satisfied, the whole chain of co-forms must be renormalised consistently as a single cohomology class.
Nonrenormalization analogy: the N = (2,2) sigma model

- An analogy to the duality invariance requirement for counter-Lagrangian integrands can be found for N=(2,2) D=2 non-linear sigma models, based on chiral superfields $T^a$.

- The sigma-model action is given by the full superspace integral
  \[ I_{\sigma} = \int d^2 x d^4 \theta K(T^a, \bar{T}^b) \]
  where the Lagrangian integrand $K(T^a, \bar{T}^b)$ is the target-space Kähler potential.

- Although the classical superspace Lagrangian integrand is not itself a globally defined scalar, the N=(2,2) nonrenormalization theorem requires all counterterms
  \[ I_{\sigma}^{ct} = \int d^2 x d^4 \theta S(T^a, \bar{T}^b) \]
  to have integrands $S(T^a, \bar{T}^b)$ that are globally defined scalars.

- Sigma models with isometries thus require counter-Lagrangian integrands that are isometrically invariant.
Off-shell half-maximal supergravity

- From the point of view of field-theoretic nonrenormalization theorems, the key question is whether there exists an off-shell linearly realised formulation of half-maximal supergravity. If so, then the nonrenormalization theorem would require a full-superspace $\int d^{16}\theta$ integral with a duality-invariant integrand, thus ruling out the F/D marginal $D=4$ and $D=5$ $R^4$ counterterms.

- Unfortunately, the answer to this question is not currently known. But there is a closely related off-shell formulation for linearized $D=10$, $N=1$ supergravity, with a finite number of component fields:

$$\mathcal{L}_{10} = \frac{1}{2} V_{abc} \Delta_{abc,def} V_{def} - V_{abc} \bar{D}\Gamma_{abc} DS \quad \Delta : D^{16}, \partial D^{14}, \text{etc.}$$

- Upon dimensional reduction to $D=4$, the $N=1$, $D=10$ theory yields $D=4$, $N=4$ supergravity plus 6 $N=4$ super-Maxwell multiplets. So one has something close to the required formalism. Pure $N=4$ SG undoubtedly would require a harmonic superspace formulation.
Current outlook

- So far, things are under control for maximal supergravity from a purely field-theoretic analysis: what is prohibited does not occur, and what is not prohibited has occurred, as far as one can see.

- As far as one knows, the first acceptable $D=4$ counterterm for maximal supergravity still occurs at $L=7$ loops ($\Delta = 16$); if not that, then they clearly exist at $L=8$ loops ($\Delta = 18$) and beyond.

- The current divergence expectations for maximal supergravity are consequently:

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Blue: known divergences

Green: anticipated divergences