6. Lagrangian Mechanics

Goal: new way of thinking about physics

old way: \( \text{Force} \rightarrow \text{EoM} \)
new way: \( \text{Action} \rightarrow \text{EoM} \)

\[ \text{Force} \rightarrow \]

Advantages: more fundamental
- clearer for conservation laws
- involves only 1st derivatives
- easier
- easier to use in other branches of physics

EoM
QFT
GR

6.1 Euler-Lagrange Equations

System: particle with position \( x \), conservative systems

\[ T = \frac{1}{2} m x^2 \quad \text{kinetic energy} \]

\[ V = V(x) \quad \text{potential energy} \]

\[ L = T - V \quad \text{Lagrangian} \]

\[ L = \frac{1}{2} m x^2 - V(x) \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial \dot{x}} \right) \]

\[ \text{Euler-Lagrange equation} \]

\[ (m \dot{x})'' = -\frac{dV}{dx} = F \]

Newton's 2nd Law
System - particle in 3-d
\[ L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \]
\[ \frac{d}{dt} \left( \frac{\delta L}{\delta \dot{x}} \right) = \frac{dV}{dx} \]
\[ m \ddot{x} = -\nabla V \]

Ex: Spring
\[ L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \]
\[ \frac{d}{dt} \left( \frac{\delta L}{\delta \dot{x}} \right) = \frac{dL}{dx} = -kx \]
\[ m \ddot{x} \]

Principle of Stationary Action

Principle: Given a trajectory with fixed end points \( x(t_1) = x_1 \) and \( x(t_2) = x_2 \), the physical path \( x(t) \) extremizes the action

\[ S = \int L(x, \dot{x}, t) dt \]

among all paths \( x(t) \).

\[ x(t_1) \quad x_1 \]
\[ x(t_2) \quad x_2 \]
\[ x(t) \quad \bar{x}(t) \]
\[ \bar{x}(t) = x(t) + \alpha(t) \quad \text{family of curves} \]
We consider the action among the one-parameter family of curves \( x_0(t) \).

Write \[ \frac{dS}{dt} \bigg|_{t=0} = 0 \implies x_0(0) \text{ extremizes action} \]

\[
\frac{dS}{dt} = \int \left( \frac{d\dot{x}_i}{dt} \frac{dx_i}{d\alpha} + \frac{d\dot{x}_i}{d\alpha} \frac{dx_i}{dt} \right) dt
\]

\[= \int \left( \frac{dL}{dx_i} - \frac{d}{dx_i} \frac{dL}{d\dot{x}_i} \right) \frac{dx_i}{d\alpha} dt \bigg|_{t=0} = 0 \]

\[A \Rightarrow \gamma \]  

\[
\frac{dL}{dx_i} = \frac{d}{dt} \frac{dL}{d\dot{x}_i} 
\]

Note: Explain Lemma of Variational Calculus.

Names:
- Least action principle
- Stationary action principle
- Hamilton principle

Note: 1) Does the Hamilton principle mean something we

\[ \text{we don't know, namely physical laws?} \]

\[ \text{Yes and no.} \]

Given \( \dot{x}(t), \ddot{x}(t) \), then to find for which \( \ddot{x}(t) = \) determined

It says: among all paths with \( \ddot{x}(t) = x_0 \), the physical one extremizes \( S \)

\[ \dot{x}(t), \ddot{x}(t) \rightarrow \ddot{x}(t), \ddot{x}(t) \]

\[ \dot{x}(t), \ddot{x}(t) \rightarrow \ddot{x}(t), \ddot{x}(t) \]

2) \( \ddot{x} \) usually a minimum

sometimes middle of

\[ \text{never maximum} \]
\[ \Delta S = \frac{1}{2} \left( m \dot{y}^2 - k y^2 \right) dt > 0 \]

for harmonic oscillator, \( \text{soft} \) \( \text{minimum} \)

II) for light: \( S = \text{light travel time} \)

physical path

\[ \text{shortest length} \rightarrow \text{lowest} \ x^2 \rightarrow \text{smallest} \ S \]

\[ \vec{y}(t_1) - \vec{y}(t_2) \]

\( C_1 \): global minimum

\( C_2 \):

\[ \vec{y}(t_1) \]

\[ x(t_1) = \vec{v}_c(t_1) + a \vec{f}(t_1) \] shorter

\[ x(t_1) = \vec{v}_c(t_1) + a \vec{f}(t_1) \] longer

saddle

Ex. Particle on a sphere

\[ x(t) \]
Def: Given space \( M \)
Curve on \( M \) which extremizes length between two points: \( \text{geodesic} \)

Ex: \( M = S^2 \) geodesic: great circle

Change of Coordinates

\( x_1, x_m \) first set of coords. e.g. Cartesian \( x, y, z \)
\( \phi_1, \phi_m \) second set \( \pi \) e.g. polar \( \theta, \phi \)
\( L(x, \dot{x}) \) Fed. in phase space

\( S = \int L(x(q), \dot{x}(q)) \, dt \) ind. of coords. used
to

Corollary: action principle for coordinate invariant

Goal: EoM in \( q, \dot{q} \) coords.

Step 1: \( x(q), \dot{x}(q, \dot{q}) \)

Step 2: \( S = \int L(q, \dot{q}) \, dt \)

Step 3: \( \frac{d}{dt} \frac{\delta L}{\delta \dot{q}_i} = \frac{\delta L}{\delta q_i} \)

Ex: \( x = r \) and \( \theta, \phi \)
\( x = r \) and \( \theta, \phi \)
\( z = r \cos \theta \)
\[ x^2 + y^2 + z^2 = r^2 + r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \]

\[ L = \frac{1}{2} \left( r^2 + r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \right) \]

\[ \text{ELE} \quad \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \]

\[ \dot{r} = \ddot{r}^2 + r^2 \ddot{\theta} \dot{\phi}^2 \]

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**Force of Constraint**

Ex: particle on a semi-loop

![Diagram of particle on a semi-loop](Image)

1st analysis: mg \( V \)

\[ L = \frac{1}{2} mR^2 \ddot{\phi}^2 - mg R \cos \phi \]

\[ mR^2 \ddot{\phi} = mg R \sin \phi \]

\[ \ddot{\phi} = \frac{g R \sin \phi}{R} \approx \frac{g}{R} \quad \text{small } \phi \]

2nd analysis: \( r, \theta, \phi \) as variables

\[ L = \frac{1}{2} m \dot{\phi}^2 + \frac{1}{2} mR^2 \ddot{\phi}^2 - mg R \cos \phi - V(r) \]

\[ m \left( \dot{\phi}^2 \right)' = mg R \sin \phi \]

\[ m \ddot{\phi} = -V'(r) + mr^2 \dot{\theta}^2 - mg \cos \phi \]

If \( r = R \) then require constraint force

\[ -V'(R) = mg \cos \theta - mR \dot{\theta}^2 \]

\[ F'(R) \quad \text{upward force} \]
F(R) > 0  tall guys in hoop  
F(R) = 0  tall guys hoop  ![Sketch]
\( \text{good } = R^2 \)

Method:
- Start with reduced system
- Add direction II constraint force with V(r)
- Consider FDM in II direction
- Find V such that constraint is satisfied

6.2 Symmetry and Conservation Laws

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \]

Thm: If \( L \) is independent of \( q_i \) then
\[ \frac{\partial L}{\partial q_i} = p_i \]

is conserved

Note: \( q_i \) is then called a cyclic coord.

Ex: Translational invariance  \( \frac{dL}{dx} = 0 \)

in x direction
\[ \frac{d}{dx} = p_x \]

is conserved

momentum conservation

Ex: Rotational invariance in cylindrical symm.
\[ \frac{dL}{d\theta} = \frac{1}{2} m(r^2 + \dot{r}^2 + \dot{\theta}^2) - V(r) \]
\[ \frac{dL}{d\phi} = m\dot{r} \]

angular momentum
Angular momentum conservation

\[ \frac{dL}{dt} = 0 \quad \text{if} \quad V = V(r) \implies m \dot{\omega} = 0 \]

Linear momentum conservation (again)

Note: \( \frac{dL}{dt} \neq 0 \) even if \( V = 0 \).

Ex: Rotation invariance in spherical coordinates

\[ L = \frac{1}{2} (r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \quad r \neq 0 \]

\[ -V(r, \theta) \]

\[ \frac{dL}{d\phi} = 0 \implies r^2 \sin^2 \theta \dot{\phi} = \text{const} \]

Angular momentum conservation

Ex: Hamiltonian

\[ H = \sum_{i=1}^{N} \frac{\dot{q}_i^2}{2m} + \sum_{i=1}^{N} V(q_i) \]

Ex: \[ L = \frac{1}{2} m \dot{q}_i^2 + V(q_i) \]

\[ H = \frac{1}{2} m \dot{q}_i^2 + V(q_i) - E \]

Thm: \( \frac{dH}{dt} = - \frac{dL}{dt} \quad \text{Hamilton's principle} \]

Ex: \[ \dot{H} = \sum_{i=1}^{N} \left[ \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \right] \dot{q}_i - \frac{dL}{dt} \]

\[ \frac{dL}{dt} = 0 \implies \text{energy conservation} \]

Time Hamilton invariance

Ex: When this is not true
Thm. (Noether) For each symmetry of $L$ there is a conserved quantity

Grav. $q_i \rightarrow q_i + \varepsilon k_i (q_i)$

Leaver Invariant $\frac{dL}{de} = 0$

$0 - \frac{dL}{de} = \sum \left( \frac{dL}{dq_i} \frac{dq_i}{de} + \frac{dL}{dk_i} \frac{dk_i}{de} \right)$

$= \sum \left( \frac{dL}{dq_i} \frac{dq_i}{de} + \frac{dL}{dk_i} \frac{dk_i}{de} \right)$

$= \sum \left( \frac{d}{dq_i} \left( \frac{dL}{dq_i} \right) k_i + \frac{d}{dk_i} \frac{dL}{dk_i} k_i \right)$

$= \frac{d}{dt} \left( \sum \frac{d}{dq_i} \frac{dL}{dq_i} k_i \right)$

$= \sum \frac{d}{dq_i} \frac{dL}{dq_i} k_i$

$=$ conserved quantity

Ex. Translation invariance $q_i \rightarrow q_i + \varepsilon v_i$

$L(q, \dot{q}) = \sum \frac{d}{dq_i} \frac{dL}{dq_i} v_i$

momentum in direction $\vec{v}$

Ex.

$L = \frac{1}{2} (x^2 + y^2) - \frac{k}{2} (x^2 + y^2)$

$x = x + \varepsilon x \rightarrow k_x = 0$

$y = y + \varepsilon y \rightarrow k_y = 0$

$\frac{dL}{de} |_{\varepsilon = 0} = 0$

$p = xy - yx = z$ component of angular momentum
Ex
\[ L = \frac{1}{2} Mr^2 + \frac{1}{2} m (r^2 + 2\bar{r}^2) + Mg (l - r) \]

\[ l - r = \bar{r} \]

Goal: 
\text{fixed point}

\text{small oscillation about fixed point}

\text{step 1:}
\[ (mr^2 \ddot{\theta}) = 0 \]
\[ (M + m) r'' = mr \dot{\theta}^2 - Mg = \frac{L^2}{m\bar{r}^3} - Mg \]
\[ \text{fixed point: } r_0^3 = \frac{L^2}{m\bar{r}^3} \]
\[ r_0 = \frac{M\bar{r}}{m\bar{r}^2} \]

\text{step 2: a) }
\[ r = r_0 + \delta r \]
\[ \theta = \bar{\theta} + \delta \theta \]
\[ \text{expand } L \text{ to quadratic order in } \delta r, \delta \theta \]
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} \]
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \]
\[ \frac{dL}{d\delta r} = 0 \]
\[ \frac{dL}{d\delta \theta} \]

\text{complicated}

Use Lagrange's
\[ \delta L = L = 0 = L_0 \text{ (zero)} \]

\[ L = \frac{1}{2} (M + m) \dot{r}^2 + \frac{1}{2} \frac{L^2}{m\bar{r}^2} + Mg (l - r) \]

\[ = \frac{1}{2} (M + m) \dot{r}_0^2 + (M + m) \int_0^r \delta r^2 + \frac{1}{2} (M + m) \delta r^2 \]
\[ + \frac{1}{2} \frac{L^2}{m \tau_0^2} \left( \frac{d^2 \delta \tau}{dt^2} + 3 \frac{dr^2}{\tau_0^2} \right) - M g \delta \tau \]

Constant terms: irrelevant.
Linear terms cancel.

\[(M+m) \delta \tau'' = - \frac{3L^2}{m \tau_0^2} \delta \tau\]

\[\delta \tau'' + \frac{3L^2}{m(M+m) \tau_0^2} \delta \tau = 0\]

\[W = \left( \frac{3L^2}{m(M+m)} \right)^{1/2}\]

Recipe:
- Consider all conservation laws.
- Eliminate EM to reduce DOF.
- Find fixed point.
- Expand \( A(\varepsilon^2) \) about fixed point.

Example: Minimal surface of revolution.

\[ A = \int_0^L x \sqrt{1+y'^2} \, dx \]

\[ \delta A = 0 \text{ among all } y(x) \]

\[ \frac{d}{dx} \left( \frac{J_1(J_0 + y'^2)}{y''} \right) = \frac{J_1}{y''} \]

\[ \left( \frac{1}{1+y'^2} \right)' = \frac{1+y'^2}{y''} \]

\[ \frac{y''}{y'^2} - \frac{yy'''}{y'^3} + \frac{y'''}{y'^2} = \frac{y''}{y'} \]

\[ y y'' (1+y'^2 - y'^2) + y^2 (1+y'^2) = (1+y'^2)^2 \]

\[ y y'' = 1+y'^2 \] (nonlinear)!!
\[ y = e^{\alpha x} \quad \text{is a solution} \]

\[ \frac{1 + y'^2}{1 + y^2} = \frac{y'}{y} \quad \text{eq} \]

\[ \frac{1}{2} \ln(1 + y'^2) = by^2 + C \quad \text{eq} \]

\[ 1 + y'^2 = 2y^2 \quad B = e^C \]

\[ y(x) = \frac{1}{1B} e^{\alpha h[1B(x + t)]} \]

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**ExMotion of Test Particle**

\[ L = \frac{1}{2} \left( \frac{\dot{x}^2}{a^2} - \frac{\dot{x}^2}{\Xi^2} \right) = \frac{1}{2} \eta_{\mu \nu} \dot{x}^\mu \dot{x}^\nu = \frac{a^2}{\Xi^2} \]

\[ \dot{x}^\nu = \left(\begin{array}{c} \dot{t} \\ \dot{x}^1 \\ \dot{x}^2 \end{array}\right) \quad \eta_{\mu \nu} = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array}\right) \]

\[ S = \int L \, dt \]

\[ SS = \int dS = \int a^2 \eta_{\mu \nu} \dot{x}^\mu \dot{x}^\nu = -\int dV \eta_{\mu \nu} \dot{x}^\mu \dot{x}^\nu = 0 \]

\[ \Rightarrow \dot{x}^\mu = 0 \]

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**ExMotion of Test Particle to Curved \( S - T \)**

\[ \eta_{\mu \nu} \rightarrow \eta_{\mu \nu}(x, t) \]

\[ SS = \int dS \left( \eta_{\mu \nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} \frac{d}{dx^\mu} \eta_{\mu \nu} \dot{x}^\nu \dot{x}^\nu dx^\mu \right) \]

\[ = -\int dS \left[ \eta_{\mu \nu} \dot{x}^\mu \dot{x}^\nu \right] - \frac{1}{2} \frac{d}{dx^\nu} \eta_{\mu \nu} \dot{x}^\mu \dot{x}^\nu \right] dx^\nu = 0 \]

\[ \Rightarrow \left( \eta_{\mu \nu} \dot{x}^\mu \dot{x}^\nu \right) - \frac{1}{2} \frac{d}{dx^\nu} \eta_{\mu \nu} \dot{x}^\mu \dot{x}^\nu \right] dx^\nu = 0 \]

geodesic eq. in curved \( S - T \)

Del : \text{geodesic}