1. Dynamic, Part I

1.1 Newton's Laws

N1 1st law \( v = \text{const} \) for body if \( F = 0 \)

N2 2nd law \( \frac{dv}{dt} = F \) Note: other \( p = m v \)

N3 3rd law \( F_1 = -F_2 \)

\[ \leftrightarrow \]

\( B_1 \quad F_1, \quad B_2 \quad F_2 \)

Newton's law not valid in all coord. systems ("frames")

Initial frame: Frame in which Newton's eq. are valid

A: What is an initial frame?

Newton's absolute space

HW1: 1) What is absolute space?

4) Is concept of absolute space consistent with N3?

HW2: F2 moves uniformly w.r.t. F2

F1 install \( \rightarrow \) F2 install

A: What is absolute space?
1. 1st law: universality
   Every particle has the same inertial frame

2. 2nd law: universal response to any force
   \[ a = \frac{F}{m} \]
   \[ F = m a \] is a vector equation

3. System of point particles with fixed masses
   3rd law: total momentum conserved
   \[ p_i = m_i v_i \]
   \[ i = \text{index, refer to ith particle} \]

\[ F_{12} = F_{21} \]

\[ F_{12} = G \frac{m_1 m_2}{r^2} = F_{21} \]

\[ m_1 = G \frac{m_2}{r^2} \]

\[ m_1 \] gravitational mass

\[ m_2 \] producer force field

\[ \text{sketch} \]
Galileo: universality of gravity

[sketch] feather/stone

\[ \frac{m_1 a_1}{m_2} = G \frac{m_1 m_2}{r^2} \]

universality \( \rightarrow \) \( m^F = m^G \)

Q: why?

A: lead to Einstein's GR

Q: why a mystery?

A: very different from EM

\[ F = \frac{q_1 q_2}{r^2} \] charge

\[ 0 \quad \text{proton} \quad 0 \quad \text{neutron} \]

\[ 0 \quad \text{charge 1} \quad 0 \quad \text{charge 0} \]

same mass

different E field

different E force
1.2 Dynamic Systems

Consider particle moving in 1d

\[ F = ma \]

\[ \dot{q} = \frac{dq}{dt} \text{ velocity} \]

\[ a = \frac{d^2 q}{dt^2} \text{ acceleration} \]

\[ F = F(q, \dot{q}, t) \]

(E1) \[ m \frac{d^2 q}{dt^2} = F(q, \dot{q}, t) \] ordinary differential equation second order dynamical system

Questions:
- Who has not seen DE?  
- Are linear DE easy?  
- Is linear DE hard?  
- Can we find exact solutions?  

Example: spring q = displacement  

\[ F_{q1} = -kq \]

\[ m \ddot{q} = -kq \]  

Simple DE: solution exponential  

d = \text{exp}  

d = \text{power law}  

Analyze: \[ q(t) = e^{\frac{w^2}{2m}t} \]

\[ m \frac{d^2 q}{dt^2} = -k \quad d = -\left( \frac{k}{m} \right) \Rightarrow d = \pm iw \]
Ex: \( F(q, t) = +k t^{-2} \quad q \)
\[ m \ddot{q} = +k t^{-2} q \]

exponential: not a solution

parabolic: \( q(t) = t^a \)
\[ m \dot{d}^2 = +k \quad d = \pm \left( \frac{1}{m} \right)^{1/2} \]

Def: dynamical system autonomous if \( \frac{dF}{dt} = 0 \)

Q: Who have not seen partial derivatives?

Def: \( p = \hbar q \) momentum

Second order DE (E1) can be written as a pair of coupled first order DE
\[
\begin{align*}
\dot{q} &= \frac{p}{m} \\
p &= F(q, p, t)
\end{align*}
\]

\[ x = (q, p) \text{ phase space} \in \mathbb{R}^2 \in \mathbb{R}^{2n} \]

\[ q \text{ configuration space} \in \mathbb{R} \in \mathbb{R}^n \quad q_i \]

\[ p \text{ momentum space} \in \mathbb{R} \in \mathbb{R}^n \quad p_i \]

\[ n = 3 \quad 1 \text{ particle in 3d} \]
\[ 3 \text{ particles in 1d} \]

\[ m > 3 \quad n \text{ particles in 1d} \]
\[ \forall m \neq 3 \text{ in } 3d \]
Consider a particle in 1-d of mass $m_i$ and $\mathbf{F}_i = (F_1, \ldots, F_n)$

$$\dot{q}_i = \frac{F_i}{m_i}$$

$$\dot{p}_i = F_i q_i - g(q_i)$$

E.g. grav. force between 2 particles

**Flow in phase space**

Ex spring $q = \frac{p}{m}$

$$\dot{p} = -kq$$

Start at rest

Start at equilibrium with push

Resulting orbit

Ex $m = k = 1$ $\dot{q} = p$ $\dot{p} = -q$

Procedure:
- Take grid of points in phase space
- At each pt, draw tangent vector
- Trajectory from any starting pt.
  - Fllows a trajectory which at all times is parallel to $\mathbf{F}$ vectors
Consider initial time $s$, initial phase space point $x_0$. Evolve (E2) with time $t > s$:

$$x = \phi_{t-s}(x)$$

$\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ is the flow of a dynamical system.

Note: If $F$ is autonomous, then $\phi_{t-s} = \phi_{t-s}$.

Local existence and Global uniqueness \( \phi(t) \) solution of (E2).

Then if $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and differentiable,

- for all $s \in \mathbb{R}$
- for all $y \in \mathbb{R}^2$
- for all $U$ innbh $y$ in $\mathbb{R}^2$

such that flow exists and is unique locally in time (J) and phase space (U).

$$x = \phi_{t-s}(y) \text{ exists } \forall t \in J \forall x \in U$$

and

i) $\phi_0(y) = y$

ii) $\frac{d}{dt} \phi_{t-s}(y) = F(\phi_{t-s}(y))$

iii) $\phi_{t-s}(t)$ is $C^1$ in $t, y$

Note: Theorem generalizes to many d.o.f. systems.
Ex. Spring \[ \dot{q} = p \]
\[ p = -q \] (ps sketch)

Ex. "Double well" \[ F(q) = -4q(q^2 - 1) \]

Spring: \[ F = 0 \text{ at } q = 0 \]

Well: \[ F = 0 \text{ at } q = 0, \pm 1 \]

\[ q \]
\[ \downarrow \]
\[ \uparrow \]
\[ \downarrow \]
\[ \uparrow \]
\[ \downarrow \]
\[ \uparrow \]

\[ \text{about } (q) = (0) \]

\[ \text{about } (q) = (0) \rightarrow F = -(q^2 - 1) \]

\[ \text{about } (q) = (-1) \text{ symmetry} \]

Interpretation: gravity on surface of earth
\[ V(q) = q^2 \]
\[ F = -\frac{dV}{dq} \]

\[ V = (q-1)(q+1)^2 \text{ double well} \]
Ex: system with friction

\[ \dot{q} = -\gamma \dot{p} - q \quad 0 < \gamma \ll 1 \]

\[ \ddot{q} = \pm \gamma q - q \quad 0 < \gamma \ll 1 \]

Ex: system with antifriction

\[ \dot{p} = -\gamma \dot{q} - q \]

\[ \ddot{p} = \gamma p - q \]

Stability

\[ \text{DUE: \ if \ } \int_{t_0}^{t_1} (\dot{q})^2 \, dt \leq \frac{\gamma}{\gamma + 1} \]

Ex: spring: \[ (\dot{q}) = (0) \] \text{ is fixed } \dot{p}.

\[ (\dot{p}) = (0) \]

double well: \[ (\dot{q}) = (0, 0, 0) \]

unstable \quad stable
Mathematical description of stability:

**Phase space vector**

\[ \| \mathbf{v} \| \text{ length of vector in } \mathbb{R}^2 \]

**Def:** A point \( x_0 \) in phase space is a stable fixed point under the flow \( \phi_{t,s} \) if

\[ \forall \varepsilon > 0 \\exists \delta > 0 \text{ such that if } \| x_0 \| < \delta \]

\[ \| \phi_{t,s}(x) - x_0 \| < \varepsilon \text{ for } t > s, s > 0 \]

- Spring: \((0)\) stable
- Double well: \((0)\) unstable

N.B. give the arguments

**Potential**

a) particle in 1 d, position \( q \)

\[ F(q), \]

\[ v = \dot{q}, \]

\[ m \frac{d}{dt} \dot{q} = F, \]

\[ m \frac{d^2 q}{dt^2} = \frac{d}{dq} m \frac{dv}{dt} \]

Consider motion from 0 to \( q \)

\[ m \int dq \frac{dv}{dq} = \int F dq \]

\[ \frac{1}{2} m \int dq \left( \frac{dv}{dq} \right)^2 = \frac{1}{2} m v^2 \]

\[ q_0 = E(q) - E_k(q_0) \]
Ex: Spring
\[ F = -kx \]
\[ V = \frac{1}{2} kx^2 \]

Kinetic energy
\[ E_k = \frac{1}{2} mv^2 \]

Potential energy
\[ V(q_1) = -\int F (q) dq' \text{ potential energy} \]

relative to \( q = 0 \)

Note: \( F(q) = -\frac{dV}{dq} \)

Note: \( F = F(q) \Rightarrow E_{\text{total}} = E_k + V \text{ conserved} \)

4) n: particles with central force \( (m \text{ 3 Li}) \)

\[ F_i = \frac{m_i}{m} \sum_{i=1}^{n} \frac{F_{ij} (q_i - q_j)}{r_{ij}} = F_i \]

\[ F_{ij} (q_i, q_j) = \frac{q_i - q_j}{r_{ij}^3} \]

central force system

\[ F_{ij} = \text{Newtonian gravity} \]

\[ \rightarrow ~ \Rightarrow \leq 0 \]

\[ \frac{m_1}{m} \]

\[ f_{12} = f_{21} = -G \frac{m_1 m_2}{r^2}, \quad r = |q_1 - q_2| \]

Theorem: For an n-particle system with central forces \( F_i \) that create a potential function \( V(q_1, q_m) \) such that

\[ F_i = -\frac{\partial V}{\partial q_i} \]

proved by construction
\[ V = \sum_{1 \leq i < j \leq n} V_{ij} (q_i - q_j) - \int dr f_{ij} (r) \]

\[ \frac{3}{4 \pi} \sum_{1 \leq i < j \leq n} V_{ij} (q_i - q_j) = \delta_{ij} (q_i - q_j) \frac{q_i - q_j}{|q_i - q_j|} \]

Ex 2 particles connected by springs

\[ \begin{array}{c}
  \begin{array}{c}
    q_1 \rightarrow d \\
    q_2 \rightarrow 2d
  \end{array}
  \\
  \end{array} \]

\[ V(q_1, q_2) = \frac{1}{2} k q_1^2 + \frac{1}{2} k (q_2 - q_1)^2 + \frac{1}{2} k (d - q_2)^2 \]

\[ m_1 \ddot{q}_1 = -\frac{d}{dq_1} V = -k q_1 + k (q_2 - q_1) \]

\[ m_2 \ddot{q}_2 = -\frac{d}{dq_2} V = -k (q_2 - q_1) + k (d - q_2) \]

stable fixed point

\[ q_2 = 2q_1 \]

\[ q_1 = \frac{1}{3} d \]

\[ q_2 = \frac{2}{3} d \]

Linearization

Ex double well

\( F(q) = -4q (q^2 - 1) = -4q (q + 1)(q - 1) \)

sketch of plot

focus \( d \times \) near \( (0) \)

\[ q = \hat{q} + 1 \]

\[ |q| \ll 1 \]

\[ F(q) = -4(1 + \hat{q}) \hat{q} (2 + \hat{q}) = -8\hat{q} + O(\hat{q}^2) \]

\[ \ddot{q} = -8\hat{q} \]
solution: $\hat{q}(t) = A\sin(\sqrt{8}t) + B\cos(\sqrt{8}t)$

$|A|/|B| \ll 1$

Procedure: * find fixed point $(q_0)$

* expand about fixed point $(\hat{q}) = (q_0 + \hat{q})$

* Taylor expand force to linear order in $\hat{q}$ about $q_0$

Express well, expand about unstable fixed pt. $(\bar{q})$

$(\hat{q}) - (q) = \begin{vmatrix} 1 \end{vmatrix} \ll 1$

$F(q) = + 4q + O(q^2)$

$\bar{q} = 4q$

$q_{AB} = Ae^{-2t} + Be^{-2t}$

$|A|, |B| \ll 1$

Linearization breaks down once $|2t| \gg 1$

Hooke's Law: Any dynamical system can be linearized about a stable fixed point and solutions of linearized equations is good approximation for all times.

Procedure: at the level of potential

* find fixed point $(q_0)$

* expand about fixed pt. $(\hat{q}) = (q_0 + \hat{q})$
* Taylor expand $V$ to 2nd order in $\hat{q}$

**Ex:**

$$V(q) = (q-1)^2(q+1)^2$$

$$q_0 = 1 \quad q = 1 + \hat{q}$$

$$V(q) = \hat{q}^2(2+\hat{q})^2 = 4 \hat{q}^2 + O(\hat{q}^3)$$

$$F(q) = -\frac{dV}{dq} = -8\hat{q}$$

$$\hat{q} = -8\hat{q} \quad \text{as before}$$