Problem 1

(a) The Lagrangian density is

$$\mathcal{L} = \sqrt{-g} \left( -\frac{1}{4} F^\mu{}_{\nu} F^\nu{}_{\mu} + A^\mu J^\mu \right)$$

and we're told to ignore the $A^\mu J^\mu$ part.

We have that the energy-momentum tensor obeys

$$T^\mu{}_{\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^\mu{}_{\nu}}$$

with

$$S^\mu{}_{\nu} = \int d^4x \sqrt{-g} \left( -\frac{1}{4} g^\mu{}_{\omega} g^\nu{}_{\chi} F^\omega{}_{\chi} F^\nu{}_{\mu} \right).$$

Looking at perturbations with respect to the metric we get that

$$\delta S^\mu{}_{\nu} = \int d^4x \left( -\frac{1}{4} F^\mu{}_{\nu} F^\nu{}_{\mu} \right) S \sqrt{-g}$$

$$+ \int d^4x \sqrt{-g} \left( -\frac{1}{4} F^\mu{}_{\nu} F^\nu{}_{\mu} \right) \left( \delta g^\mu{}_{\omega} g^\nu{}_{\chi} + g^\mu{}_{\omega} \delta g^\nu{}_{\chi} \right).$$
\[ \int \text{d}^4 x \sqrt{-g} \left( -\frac{1}{4} \left( -\frac{1}{2} \right) F_{\mu \nu} F^{\mu \nu} g^{\alpha \beta} \delta g^{\alpha \beta} \\
+ \delta g^{\alpha \beta} g^{\mu \nu} \left( -\frac{1}{2} \right) F_{\beta \gamma} F_{\mu \nu} \right) \\
+ \delta g^{\alpha \beta} g^{\mu \nu} \left( -\frac{1}{2} \right) F_{\omega \beta} F_{\mu \nu} \right) \]

where we used Eq. 4.69 in the textbook to rewrite the first line.

Thus
\[ \delta S_M = \int \text{d}^4 x \sqrt{-g} \delta g^{\alpha \beta} \]
\[ \times \left( \epsilon^{\alpha \beta \gamma \delta} \frac{1}{8} F_{\mu \nu} F_{\delta \gamma} - \frac{1}{2} F_{\alpha \nu} F_{\beta \gamma} \right). \]

Thus
\[ T^{\alpha \beta} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\alpha \beta}} = F_{\alpha \nu} F_{\beta \gamma} - \frac{1}{4} g^{\alpha \beta} F_{\mu \gamma} F_{\delta \nu}. \]

(b) Now the Lagrangian becomes
\[ \mathcal{L}_M = \sqrt{-g} \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + A_{\nu} J^{\nu} + \beta R_{\mu \nu} g^{\rho \sigma} F_{\mu \rho} F_{\nu \sigma} \right). \]

(Note that the new term also has a factor \( \sqrt{-g} \); otherwise \( S_M = \int \text{d}^4 x \mathcal{L} \) wouldn't be a scalar.)
Let's first derive Maxwell's equations.

We write \( \mathcal{F}_m = \sqrt{-g} \mathcal{F}_m \). Then the Euler–Lagrange equation is

\[
\frac{\partial \mathcal{F}_m}{\partial A_y} - \nabla^m \left( \frac{\partial \mathcal{F}_m}{\partial (\nabla^m A_y)} \right) = 0
\]

(see Eq. 4.49 in the textbook).

It's easy to see that

\[
\frac{\partial \mathcal{F}_m}{\partial A_y} = J^y.
\]

Furthermore, since

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu
\]

we get that

\[
\frac{\partial F_{\mu\nu}}{\partial (\nabla_\alpha A_\beta)} = 2 F_{\mu\nu} \frac{\partial F_{\mu\nu}}{\partial (\nabla_\alpha A_\beta)}
\]

\[
= 2 F_{\mu\nu} \left( \delta_\mu^\alpha \delta_\nu^\beta - \delta_\beta^\mu \delta_\alpha^\nu \right)
\]

\[
= 2 F_{\alpha\beta} - 2 F_{\beta\alpha} = 4 F_{\alpha\beta}
\]

and
\[
\frac{\partial F_{\mu \nu} F_{\rho \sigma}}{\partial (\nabla_\alpha A_\beta)} = (\delta_\mu^\chi \delta_\rho^\beta - \delta_\rho^\chi \delta_\mu^\beta) F_{\chi \sigma} \\
+ F_{\mu \rho} (\delta_\nu^\chi \delta_\sigma^\beta - \delta_\sigma^\chi \delta_\nu^\beta)
\]

so that

\[
\frac{\partial (\beta R^{\mu \nu} g_{\rho \sigma} F_{\mu \nu} F_{\rho \sigma})}{\partial (\nabla_\alpha A_\rho)}
\]

\[
= \beta (R^{\chi \nu} g_{\rho \sigma} - R^{\beta \nu} g_{\chi \sigma}) F_{\rho \sigma} F_{\chi \nu}
\]

\[
+ \beta (R^{\chi \chi} g_{\rho \sigma} - R^{\rho \sigma} g_{\chi \chi}) F_{\mu \rho}
\]

\[
= 2\beta (R^{\chi \nu} g_{\beta \sigma} - R^{\beta \nu} g_{\chi \sigma}) F_{\rho \sigma}
\]

\[
= 2\beta (R^{\chi \nu} F_{\chi \beta} - R^{\beta \nu} F_{\chi \chi})
\]

where we've used that \(R^{\chi \chi} = R^{\chi \nu}\).

Bringing all the pieces together gives

\[
\nabla_\alpha (F_{\chi \beta} + \beta (R^{\chi \nu} F_{\nu \beta} - R^{\beta \nu} F_{\chi \chi})) - \Box \beta = 0.
\]

This can be written as
\[ \nabla_\alpha F^{\beta \alpha} + 2 \beta \nabla_\alpha (R^{\kappa \nu} F_{\nu \beta} - R^{\beta \nu} F_{\nu \kappa}) = J^\beta \]

When \( \beta \to 0 \) this reduces to Maxwell's equations.

Let's now check that the current is conserved, i.e. \( \nabla_\beta J^\beta = 0 \).

This is equivalent to showing that

\[ \nabla_\beta \nabla_\alpha F^{\beta \alpha} + 2 \beta \nabla_\beta \nabla_\alpha (R^{\kappa \nu} F_{\nu \beta} - R^{\beta \nu} F_{\nu \kappa}) = 0. \]

This is easy to see from the fact that both \( F^{\beta \alpha} \) and \( R^{\kappa \nu} F_{\nu \beta} - R^{\beta \nu} F_{\nu \kappa} \) are antisymmetric under interchange of \( \kappa, \beta \).

For any tensor \( B^{\alpha \beta} \) so that \( B^{\alpha \beta} = -B^{\beta \alpha} \) we have that
\[ \nabla_\alpha \nabla_\beta B^{\alpha \beta} \]
\[ = \frac{1}{2} \nabla_\alpha \nabla_\beta \left( B^{\alpha \beta} - B^{\beta \alpha} \right) \]
\[ = \frac{1}{2} \left( \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha \right) B^{\alpha \beta} \]

in second term

\[ = \frac{1}{2} R^\alpha {}_{\lambda \beta \gamma} B^{\lambda \beta} + \frac{1}{2} R^\beta {}_{\lambda \alpha \gamma} B^{\alpha \lambda} \]
\[ = - R^\beta {}_{\lambda \beta \alpha} \]

see eq. 3.114 in textbook

\[ = \frac{1}{2} \left( R_{\lambda \beta} B^{\lambda \beta} - R_{\lambda \alpha} B^{\alpha \lambda} \right) \]
\[ = R_{\lambda \alpha} B^{\lambda \alpha} \]
\[ = 0 \]

because \( R_{\lambda \alpha} \) is symmetric but \( B_{\lambda \alpha} \) is antisymmetric.
The only thing left is to show how Einstein's equation changes.

The total Lagrangian is

$$\mathcal{L}_{\text{tot}} = \sqrt{-g} \left( R + \beta R^\mu_\nu g^\rho_\sigma F_{\mu\rho} F_{\nu\sigma} \right) + \mathcal{L}$$

where $\mathcal{L}$ will only contribute to the stress-energy tensor.

As usual, $\int \sqrt{-g} \mathcal{L}$ gives

$$\frac{1}{\sqrt{-g}} \int \sqrt{-g} \mathcal{L}$$

$$\Rightarrow \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}.$$

We can thus focus on

$$\mathcal{L}' = \beta \int \sqrt{-g} g^{\rho_\sigma} R^\mu_\nu F_{\mu\rho} F_{\nu\sigma}.$$

Varying with respect to the metric gives

$$\delta \mathcal{L}' = (\delta \mathcal{L}')_1 + (\delta \mathcal{L}')_2 + (\delta \mathcal{L}')_3$$

where
\[(\delta S')_1 = \beta S d^4 x \sqrt{-g} \, g^{\rho \sigma} F^{\mu \nu} F_{\mu \nu} \delta R_{\mu \nu},\]

\[(\delta S')_2 = \beta S d^4 x \sqrt{-g} \, R^{\mu \nu} F_{\mu \nu} F_{\rho \sigma} \delta g^{\rho \sigma},\]

and

\[(\delta S')_3 = \beta S d^4 x g^{\rho \sigma} R^{\mu \nu} F_{\mu \nu} F_{\rho \sigma} \sqrt{-g} \]

Let's do the easy terms first.

We can leave \((\delta S')_2\) as it is. Using Eq. 4.69 in the textbook we get that

\[(\delta S')_3 = \beta S d^4 x g^{\rho \sigma} R^{\mu \nu} F_{\mu \nu} F_{\rho \sigma} \left(-\frac{1}{2}\right) \sqrt{-g} \, g_{\rho \sigma} \delta g^{\rho \sigma}.\]

In the textbook it is shown (Eq. 4.62) that

\[\delta R_{\mu \nu} = \nabla_\lambda (\delta P^{\mu \nu}_{\lambda}) - \nabla_\nu (\delta P^{\mu \lambda}_{\lambda}),\]

so

\[\delta R_{\mu \nu} = \nabla_\lambda (\delta P^{\mu \nu}_{\lambda}) - \nabla_\nu (\delta P^{\mu \lambda}_{\lambda}).\]

This gives us that
\[(\delta S')_4 = \beta g^{\nu \lambda} \sqrt{-g} \rho \sigma \Gamma^\rho_\nu \nabla^\nu \left[ \nabla_\lambda \delta \Gamma^\lambda_\nu \mu - \nabla_\nu \delta \Gamma^\nu_\lambda \mu \right] \]

\[
= \beta g^{\nu \lambda} \sqrt{-g} \rho \sigma \left[ \nabla_\nu \left( F^\rho_\mu F^\nu_\lambda \right) \delta \Gamma^\lambda_\nu \mu \right.

\[\left. - \nabla_\lambda \left( F^\rho_\mu F^\nu_\lambda \right) \delta \Gamma^\nu_\lambda \mu \right] \]

\[
= \beta g^{\nu \lambda} \sqrt{-g} \left[ \nabla_\nu \left( F^\rho_\mu F^\nu_\lambda \right) \delta \Gamma^\lambda_\nu \mu \right.

\[\left. - \nabla_\lambda \left( F^\rho_\mu F^\nu_\lambda \right) \delta \Gamma^\nu_\lambda \mu \right] \]

where we used Stoke's theorem to get rid of boundary terms.

Furthermore, Eq. 4.64 in the textbook tells us that

\[\delta \Gamma^\lambda_\nu \mu = - \frac{1}{2} \left[ g_{\nu \lambda} \nabla_\mu \delta g^{\omega \lambda} + g_{\omega \mu} \nabla_\nu \delta g^{\omega \lambda} \right. \]

\[\left. - g_{\nu \lambda} g_{\rho \beta} \nabla^\lambda \left( \delta g^{\rho \beta} \right) \right] \]

which means that

\[\delta \Gamma^\lambda_\nu \mu = - \frac{1}{2} \left[ g_{\nu \lambda} \nabla_\mu \delta g^{\omega \lambda} + g_{\omega \mu} \nabla_\nu \delta g^{\omega \lambda} \right. \]

\[\left. - g_{\nu \lambda} g_{\rho \beta} \nabla^\lambda \left( \delta g^{\rho \beta} \right) \right] \]
Since \( g_{\mu\nu} \Delta_{\mu} \Delta_{\nu} \delta g^{\alpha\beta} \)

\[
= \nabla_{\mu} (g_{\alpha\lambda} \nabla^{\nu} \delta g^{\alpha\lambda}) = \frac{1}{2} \nabla_{\mu} \delta (g_{\alpha\beta})
\]

\[= \frac{1}{2} \nabla_{\mu} \delta(h) = 0 \]

we can write

\[
\delta \Gamma_{\alpha\beta}^{\lambda} = -\frac{1}{2} [g_{\gamma\mu} \nabla_{\lambda} \delta g^{\mu\gamma} - g_{\lambda\mu} g_{\lambda\beta} \nabla_{\gamma} \delta g^{\mu\beta}] \]

Therefore

\[
(\delta S')_{\lambda} = -\frac{\beta}{2} \delta h^{\lambda} \times \Delta_{\nu} g^{\nu} \times \Delta_{\lambda} g^{\nu} \]

\[
\times \left[ \nabla_{\gamma} (F^{\mu\rho} F_{\mu\rho}) \right] \left( g_{\gamma\mu} \nabla_{\nu} \delta g^{\mu\nu} - g_{\nu\mu} g_{\gamma\beta} \nabla_{\nu} \delta g^{\mu\beta} \right)
\]

\[- \nabla_{\rho} (F^{\mu\nu} F_{\mu\nu}) \left( g_{\gamma\mu} \nabla_{\nu} \delta g^{\mu\nu} + g_{\nu\mu} \nabla_{\nu} \delta g^{\mu\nu} \right. \\
\left. - g_{\nu\mu} g_{\gamma\beta} \nabla_{\nu} (\delta g^{\mu\beta}) \right] \]
\[ \frac{\beta}{c} \int d^4x \sqrt{-g} \]
\[ \left[ \nabla_\lambda \nabla_\nu A^{\mu} \right] \gamma_\mu \delta g_{\omega \lambda} - \nabla_\nu A^{\mu} \gamma_\mu \delta g_{\omega \lambda} \\
- \nabla_\mu \nabla_\nu A^{\mu} \gamma_\nu \delta g_{\omega \lambda} - \nabla_\nu \nabla_\lambda A^{\mu} \gamma_\mu \delta g_{\omega \lambda} \\
+ \nabla^2 A^{\mu} \gamma_\mu \gamma_\nu \gamma_\rho \delta g_{\omega \lambda} \right] \]

where we used Stokes' theorem and have written \( A^{\mu} = F^{\mu \nu} F_{\nu} \).

It's easy to see that
\[ \nabla_\lambda \nabla_\nu A^{\mu} \gamma_\mu \delta g_{\omega \lambda} = \nabla_\lambda \nabla_\nu A^{\omega \nu} \delta g_{\omega \lambda} \]
\[ = \nabla_\lambda \nabla_\nu A^{\omega \nu} \delta g_{\omega \lambda} = \nabla_\nu \nabla_\lambda A^{\nu \mu} \gamma_\mu \delta g_{\omega \lambda} \]
\[ = \nabla_\mu \nabla_\nu A^{\nu \mu} \gamma_\mu \delta g_{\omega \lambda} \]

because \( A^{\mu} \) is symmetric. Thus the first and third term cancel.

Similarly, the second and fourth term are the same.
Thus,
\[(\delta S')_1 = \frac{\hbar}{2} \int d^4x \sqrt{-g} \left[ -2 \nabla_\alpha \nabla_\beta A^{\mu\nu} g_{\mu\nu} \delta g^{\alpha\beta} + \nabla^2 A^{\mu\nu} g_{\mu\nu} \delta g^{\alpha\beta} \right].\]

Summing everything up we get
\[\delta S' = \frac{\hbar}{2} \int d^4x \sqrt{-g} \delta g^{\alpha\beta} \left[ -2 \nabla_\alpha \nabla_\beta (F_{\beta\rho}^\mu F^{\nu\rho}) + \nabla^2 (F_{\beta\rho}^\mu F^{\nu\rho}) \right. \]
\[\left. + 2 R^{\mu\nu} F_{\mu\alpha} F_{\nu\beta} - g_{\alpha\beta} R^{\mu\nu} F_{\mu\rho} F^{\nu\rho} \right].\]

Therefore, Einstein's equation becomes
\[R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} - \]
\[\beta \left( - \nabla_\alpha \nabla_\beta (F_{\beta\rho}^\mu F^{\nu\rho}) + \frac{1}{2} \nabla^2 (F_{\beta\rho}^\mu F^{\nu\rho}) \right. \]
\[\left. + R^{\mu\nu} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{2} g_{\alpha\beta} R^{\mu\nu} F_{\mu\rho} F^{\nu\rho} \right) = 8\pi G T_{\alpha\beta}.\]
Problem 2

We write

\[ S = \int d^4 x \sqrt{-g} \; g^{\mu \nu} \; R_{\mu \nu} \]

where the Riemann tensor depends on a connection \( \Gamma \) that we vary separately from \( g \).

Let's first derive the equation of motion for the connection.

\[ \delta S = \int d^4 x \sqrt{-g} \; g^{\mu \nu} \; \delta R_{\mu \nu} \]

We can show exactly like in the textbook (see Eq. 4.62) that

\[ \delta R_{\mu \lambda \nu} = \nabla_\lambda (\delta R_{\nu \mu}) - \nabla_\nu (\delta R_{\lambda \mu}) \]

So

\[ \delta R_{\mu \nu} = \nabla_\rho \delta R_{\nu \rho} - \nabla_\nu \delta R_{\mu \rho} \]

(Remember: The covariant derivative involves the connection \( \Gamma \); we have a covariant derivative...
because $\delta^P$ is the difference of two
connections and thus a tensor, see
the argument leading up to 3.20 in the
text book.

Thus we get that

$$\delta S = \int \sqrt{-g} \left( \nabla_\mu \delta P^\mu_{\nu\mu} - \nabla_\nu \delta P^\nu_{\mu\mu} \right)$$

$$= \int \sqrt{-g} \left[ \nabla_\mu \left( g^{\mu\nu} \delta P^\nu_{\mu\nu} \right) - \nabla_\nu \left( g^{\mu\nu} \delta P^\nu_{\mu\nu} \right) \right]$$

$$- \int \sqrt{-g} \left( \delta P^\mu_{\nu\mu} \nabla_\nu g^{\mu\nu} - \delta P^\nu_{\mu\nu} \nabla_\mu g^{\mu\nu} \right)$$

The former line vanishes by Stokes's theorem
and we're left with

$$\delta S = -\int \sqrt{-g} \delta P^\nu_{\mu\nu} \left( \nabla_\mu g^{\mu\nu} - \delta_\mu^{\nu} \nabla_\mu g^{\mu\nu} \right)$$

Thus the equation of motion is

$$\nabla_\mu g^{\mu\nu} - \delta_\mu^{\nu} \nabla_\mu g^{\mu\nu} = 0$$

Acting on this equation with $\delta^\nu$ we get

$$0 = \nabla_\rho g^{\rho\mu} - \nabla_\rho \nabla_\mu g^{\mu\nu} = -3 \nabla_\mu g^{\mu\nu}.$$
Thus it's easy to see that
\[ \nabla_\rho g^{\mu \nu} = 0. \]

This means that our connection is metric compatible. We furthermore assumed that the connection is torsion-free.

But a torsion-free metric compatible connection can only be the Levi-Civita connection (see p. 99 in the textbook).

Let's now turn to variations in \( g \).

We have that
\[
\delta S = \int d^4 x \left( R_{\mu \nu} (P) g_{\mu \nu} \sqrt{-g} \right) \\
+ \int d^4 x \left( R_{\mu \nu} (P) \sqrt{-g} \right) \delta g^{\mu \nu}
\]
\[
= \int d^4 x \left( R_{\mu \nu} g_{\mu \nu} \sqrt{-g} \right) \\
+ \int d^4 x \sqrt{-g} R_{\mu \nu} \delta g^{\mu \nu}.
\]

where \( R \) is just the usual Riemann tensor (because our connection is the Levi-Civita one).
This is the same as the variation of

\[ S = \int du^x \, R_{\mu\nu} \, g^{\mu\nu} \, N \] \[ \tag{10} \]

with respect to \( g \) except that we've missing the term

\[ \int du^x \, N^{-1} \, g^{\mu\nu} \, dR_{\mu\nu} \]

but that can be shown to vanish (see Eq. 4.65 in the textbook and the subsequent discussion).

Thus exactly the same derivation as for the usual Hilbert action

\[ S_H = \int N^{-1} \, g \, R \, du^x \]

will lead to Einstein's equation
Problem 3

There are multiple ways to solve this problem.
Let's use that in normal coordinates the geodesics are straight lines

\[ x^M (\lambda) = \lambda a^M \]

for some fixed vectors \( a^M \).

The geodesic equation

\[ \frac{d^2 x^M}{d\lambda^2} + \Gamma^M_{\omega x} \frac{dx^\omega}{d\lambda} \frac{dx^x}{d\lambda} = 0 \]

then gives that

\[ \Gamma^M_{\omega x} (x) a^\omega a^x = 0 \]

Let's Taylor expand the equation

We have

\[ \gamma_{\mu \nu} \Gamma^M_{\omega x} (x) a^\omega a^x = 0 \]

in \( x \). Since \( \Gamma \) vanishes at the origin we have to go to first order
We have that

\[ g_{\alpha \mu} \nabla^\mu \omega^\nu = \frac{1}{2} \left( \partial_\omega g_{\nu \alpha} + \partial_x g_{\omega \alpha} - 2 \partial_\alpha g_{\omega x} \right) \]

so

\[ g_{\alpha \mu} \nabla^\mu \omega^\nu \]

\[ = \frac{1}{2} \left( g_{\nu \alpha} \omega^\nu + g_{\omega \alpha} \nu_\nu - g_{\nu \alpha} \omega^\nu \right) \nu^2 \]

\[ + O(x^2) \, . \]

This is valid along the geodesic \( x^2 = 1 \alpha^x \)

so

\[ g_{\alpha \mu} \nabla^\mu \omega^\nu (x) \alpha^\omega \alpha^x = 0 \]

gives

\[ (g_{\nu \alpha} \omega^\nu + g_{\omega \alpha} \nu_\nu - g_{\nu \alpha} \omega^\nu \nu^2 \alpha^\omega \alpha^x = 0 \]

This tells us that

\[ \sum_{(\nu,\omega,\alpha)} \left( g_{\nu \alpha} \omega^\nu + g_{\omega \alpha} \nu_\nu - g_{\nu \alpha} \omega^\nu \right) = 0 \]

\[ \uparrow \]

all symmetric combinations

or more explicitly
\[ 2 \left( g_{x,xx} + g_{x,x} + g_{x,x} \right) \\
- \left( g_{x,w,ww} + g_{w,ww} + g_{w,ww} \right) = 0. \]

We can rewrite this equation by doing \( x \rightarrow x \rightarrow w \rightarrow x \) which gives

\[ 2 \left( g_{x,w,ww} + g_{x,x} + g_{x,x} \right) \\
- \left( g_{w,ww} + g_{w,ww} + g_{w,ww} \right) = 0. \]

Taking this equation and doing \( x \rightarrow x \rightarrow w \rightarrow x \) then gives

\[ 2 \left( g_{w,xw} + g_{w,x} + g_{w,x} \right) \\
- \left( g_{x,x} + g_{x,x} + g_{x,x} \right) = 0. \]

Summing these three equations (I, II and III) then gives

\[ g_{x,x,x} + g_{x,w,wx} + g_{x,w,wx} = 0. \]
Interchanging \( x \) and \( y \) in \((1)\)
gives
\[
2(\text{g}_{yx, \omega x} + \text{g}_{yw, \alpha x} + \text{g}_{yx, \alpha \omega}) \\
- (\text{g}_{xw, \alpha x} + \text{g}_{w\omega, x\lambda} + \text{g}_{xx, \omega \nu}) = 0
\]
Eq. \((4)\) shows us that the negative terms vanish leading to
\[
\text{g}_{x\omega, \alpha \omega} + \text{g}_{x\omega, \gamma x} + \text{g}_{x\nu, x\omega} = 0 \quad (5)
\]
Comparing \((4)\) and \((5)\) we see that
\[
\text{g}_{x\omega, \alpha \gamma} = \text{g}_{x\nu, x\omega} \quad (5)
\]
We can now finally show that
\[
R_{x\beta \gamma} + R_{x\beta \alpha} = -3 \text{g}_{x\nu, \beta \delta}.
\]
In our normal coordinates
\[ R_{\rho \sigma \beta \delta} = g_{\alpha \lambda} \Pi^\lambda_{\delta \beta, \sigma} - g_{\alpha \lambda} \Pi^\lambda_{\sigma \beta, \delta} \]
\[ = \frac{1}{2} \left[ g_{\alpha \lambda, \delta \sigma} + g_{\alpha \sigma, \rho \delta} - g_{\delta \sigma, \alpha \rho} \right] \]
\[ - \frac{1}{2} \left[ g_{\alpha \lambda, \delta \sigma} + g_{\alpha \sigma, \rho \delta} - g_{\delta \sigma, \alpha \rho} \right] \]
\[ = \frac{1}{2} \left( g_{\alpha \delta, \rho \sigma} - g_{\delta \beta, \alpha \sigma} - g_{\delta \sigma, \beta \rho} + g_{\delta \rho, \alpha \sigma} \right) \]
\[ = g_{\delta \lambda, \rho \sigma} - g_{\delta \rho, \alpha \sigma} \]

where we used \( \Box \).

Thus
\[ R_{\rho \sigma \delta \beta} + R_{\rho \delta \beta \sigma} \]
\[ = g_{\delta \sigma, \rho \beta} - g_{\delta \rho, \alpha \sigma} + g_{\delta \sigma, \beta \alpha} - g_{\delta \rho, \alpha \sigma} \]
\[ = g_{\delta \lambda, \rho \beta} + g_{\delta \sigma, \beta \alpha} - 2 g_{\delta \rho, \alpha \sigma} \]
\[ = -3 g_{\delta \rho, \alpha \sigma} \]

where we used \( \Box \). Finally, using \( \Box \), we get
\[ R_{\rho \delta \beta \sigma} + R_{\rho \beta \delta \sigma} = -3 g_{\delta \rho, \alpha \sigma}. \]
Problem 4

We have that

\[ R^\alpha_{\beta\gamma\delta} = \langle dx^\alpha, R(\partial\delta, \partial\gamma) \partial\beta \rangle \]

Since the covariant derivative is metric compatible we get that

\[ \nabla_\varepsilon R^\alpha_{\beta\gamma\delta} = \langle \nabla_\varepsilon dx^\alpha, \nabla_\varepsilon R(\partial\delta, \partial\gamma) \partial\beta \rangle \]

\[ + \langle dx^\alpha, (\nabla_\varepsilon R)(\partial\delta, \partial\gamma) \partial\beta \rangle \]

\[ + \langle dx^\alpha, R(\nabla_\delta \partial\delta, \partial\gamma) \partial\beta \rangle \]

\[ + \langle dx^\alpha, R(\partial\delta, \nabla_\delta \partial\gamma) \partial\beta \rangle \]

\[ + \langle dx^\alpha, R(\partial\delta, \partial\gamma) \nabla_\delta \partial\beta \rangle \].

Working in a normal coordinate system \( \nabla_\varepsilon dx^\alpha = 0 \) and \( \nabla_\varepsilon \partial\beta = 0 \)

(because the Christoffel symbols vanish).

Thus

\[ \nabla_\varepsilon R^\alpha_{\beta\gamma\delta} = \langle dx^\alpha, \nabla_\varepsilon R(\partial\delta, \partial\gamma) \partial\beta \rangle \].
Problem 5

I will take as given that

\[ R(x, y) = -R(y, x) \]

\[ \langle R(x, y)z, u \rangle = -\langle R(x, y)u, z \rangle \]

\[ R(x, y)z + R(y, z)x + R(z, x)y = 0 \]

We then get that

\[ \langle R(x, y)z, u \rangle = -\langle R(y, z)x, u \rangle - \langle R(z, x)y, u \rangle \]

\[ = \langle R(y, z)u, x \rangle + \langle R(z, x)u, y \rangle \]

\[ - \langle R(z, u)y, x \rangle - \langle R(u, y)z, x \rangle \]

\[ - \langle R(x, u)z, y \rangle - \langle R(u, z)x, y \rangle \]

\[ = 2\langle R(z, u)x, y \rangle + \langle R(u, y)x, z \rangle + \langle R(x, u)y, z \rangle \]
\[ 2 \langle R(z, u) x, y \rangle - \langle R(y, x) u, z \rangle \]

\[ = 2 \langle R(z, u) x, y \rangle - \langle R(x, y) z, u \rangle. \]

Comparing the first and last expression we see that

\[ \langle R(x, y) z, u \rangle = \langle R(z, u) x, y \rangle. \]

**Problem 6**

We're supposed to show that

\[ \Phi = -\frac{1}{2} h_{00} \text{ is equivalent to } \]

\[ \nabla^2 \Phi = R_{00}, \text{ i.e. that } \]

\[ R_{00} = -\frac{1}{2} \nabla^2 h_{00}, \]

in the weak field and static limit.

I guess the conventions in the problem text didn't match up; it's supposed to be \[ \Phi = -\frac{1}{2} h_{00} \]
We have that

\[ R^\lambda_{\mu \nu \rho} = \partial_\lambda P^\lambda_{\mu \nu \rho} - \partial_\mu P^\lambda_{\nu \rho \lambda} + \partial_\nu P^\lambda_{\rho \lambda \mu} - \partial_\rho P^\lambda_{\lambda \mu \nu} \]

The term \( \partial_\mu P^\lambda_{\nu \rho \lambda} = 0 \) because we're working in the static limit.

Furthermore, it's easy to see that if

\[ g^{\mu \nu} = \eta^{\mu \nu} + h^{\mu \nu} \]

\( \eta^{\mu \nu} \) is flat metric.

Then \( \Pi = O(h) \) so terms with

\[ \Pi^2 = O(h^2) \]

and can be dropped in the weak field limit. We're then left with

\[ R^0_{\mu \nu} = \partial_\lambda P^\lambda_{\mu \nu} = \partial_\nu P^0_{\mu \nu}. \]
Since
\[ \Pi^i_{\mu \nu} = \frac{1}{2} \gamma^{i\kappa} (\partial_\mu h_{\kappa \nu} + \partial_\nu h_{\kappa \mu} - \partial_\kappa h_{\mu \nu}) \]
\[ = -\frac{1}{2} \partial_\mu h_{\nu \mu} \]
we get that
\[ R_{\mu \nu} = -\frac{1}{2} \partial_\mu \partial_\nu h_{\kappa \mu} = -\frac{1}{2} \Delta^2 h_{\kappa \mu}. \]